Optimal Noise Shaping for Networked Control Systems

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***Abstract*— In Networked Control Systems (NCS's) achievable performance is limited by the characteristics of the communication links used to transmit signals in the loop. In this paper we focus on ideal bit-rate limited channels, i.e., channels in which signals need to be quantized prior to transmission. We use noise shaping quantization ideas to develop a novel NCS architecture that takes quantization into account. Using linear time invariant system theoretical tools, we show how to design a noise shaping quantizer that minimizes the impact of quantization noise on loop performance, as measured by the variance of the tracking error component due to quantization. We provide explicit analytical expressions for both the optimal noise shaping quantizer parameters and the optimum achievable performance. It is also shown that the proposed NCS architecture outperforms other schemes recently proposed in the literature.**

I. Introduction

Standard control theory assumes that the interconnection of plant and controller is *transparent*, i.e., transmitted signals are equal to received signals. This paradigm is often appropriate and underlies many successful control design methods, as discussed, for example, in [1], [2]. However, in some situations the characteristics of the underlying communication channels, renders the assumption of ideal communication links unacceptable. Control systems where the communication link constitutes a bottleneck in achievable performance are commonly referred to as *Networked Control Systems* (NCS's); see, e.g., [3]–[6] and the references therein. The communication link can either be dedicated or consist of a network which is shared between several users. Novel aspects introduced by the presence of non-transparent communication links in control include time delays, data-dropouts and quantization [7], [8]. Moreover, from an analysis perspective, even basic system theoretic notions, such as closed loop stability and asymptotic tracking are far from trivial in the networked control context; see, e.g., [9]–[14].

When designing NCS's, the characteristics of the communication system should be explicitly taken into account to ensure acceptable performance levels. This raises new challenges [15], [16]. A key observation is that, in NCS's, there exist additional degrees of freedom in the design process as compared with traditional control loops. As a consequence, to optimize performance, it is useful to investigate architectural issues and signal coding methods; see also [17]–[19].

Several NCS architectures have been studied in the literature. One can distinguish configurations where the channel is located in the *up-link*, i.e., between sensors and controller input [9], [17], and where it lies in the *down-link*, i.e., between controller output and actuators [12]. More general architectures, where the processing power is distributed, have also been examined; see, e.g., [15], [16], [18], [20].

In this paper we assume that a given design (the *nominal* design) has been carried out under the assumption of ideal communication links, but the control loop has to be implemented considering a bit-rate limited channel in the down-link. Thus, the controller output must be quantized prior to transmission. To that end, we borrow ideas from the $\Sigma\Delta$-converter literature (see, e.g., [21], [22]), and employ noise shaping quantizers to code the controller output. We show how to design the noise shaping quantizer so as to minimize the impact of quantization noise on the loop tracking error, as measured by the variance of the tracking error component due to quantization noise. As in other contemporary approaches to NCS design, see [11], [23], [24], we will deploy design methodologies that utilise LTI system theoretic ideas.

The present paper extends work described in three early papers by the same authors [25]–[27]. Indeed, the architectures considered in those papers turn out to be special cases of the scheme considered in this paper. Consequently, the best achievable performance of NCS schemes studied in [25], [27] is never better than the best achievable performance of the noise shaping NCS's to be studied in the current paper.

The remainder of this paper is organized as follows: Section II presents technical preliminaries and definitions. Section III defines the problem of interest. Section IV presents the main results, and Section V illustrates them with a simple example. Finally, conclusions are drawn in Section VI.

II. Preliminaries and Notation

In the remainder of this paper we use standard vector space notation for signals. For example, x denotes $\{x(k)\}_{k\in\mathbb{N}_0}$. We also use z as both the argument of the z-transform and as the forward shift operator, where the meaning is clear from the context.

The set of all scalar real rational discrete time transfer functions will be denoted by $\mathcal{R}$. We also define $\mathcal{RH}_2$ as the subset of $\mathcal{R}$ composed of all strictly proper and stable transfer functions, $\mathcal{RH}_\infty$ as the subset of $\mathcal{R}$ composed of all proper and stable transfer functions, and $\mathcal{RH}_2^\perp \triangleq \mathcal{R} - \mathcal{RH}_2$ [28], [29]. We note that every $H(z) \in \mathcal{RH}_\infty$ can be decomposed as $K + \hat{H}(z)$, with $\hat{H}(z) \in \mathcal{RH}_2$ and $K \in \mathbb{R}$. If $K = 0$, then $H(z) \in \mathcal{RH}_2$ and *vice versa*.

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For every $H(z) \in \mathcal{R}$ we define the $2-$norm as

$$||H(z)||_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega}, \quad (1)$$

provided the integral converges. In particular, it converges for all $H(z) \in \mathcal{R}$ having no poles on the unit circle.

Using a partial fraction expansion, every $H(z) \in \mathcal{R}$ can be written as

$$H(z) = H_{\perp}(z) + H_2(z), \quad (2)$$

where $H_{\perp}(z) \in \mathcal{RH}_2^{\perp}$ and $H_2(z) \in \mathcal{RH}_2$. It can also be shown that [29]

$$||H(z)||_2^2 = ||H_{\perp}(z)||_2^2 + ||H_2(z)||_2^2 . \quad (3)$$

Given $H(z) \in \mathcal{R}$, we define a generalized Blaschke product for $H(z)$ as any function $\xi_H(z) \in \mathcal{R}$ such that $\xi_H(z)H(z)$ is stable, minimum phase (MP), biproper and has unit magnitude for all $z = e^{j\omega}$. Such a function exists if and only if $H(z)$ does not have poles or zeros on the unit circle. A particular expression for $\xi_H(z)$ is given by

$$\xi_H(z) = z^{\mathrm{reldg}\{H(z)\}} \left(\prod_{i=1}^{n_c} \frac{1 - z\bar{c}_i}{z - c_i}\right) \left(\prod_{j=1}^{n_p} \frac{z - p_j}{1 - z\bar{p}_j}\right), \quad (4)$$

where $\mathrm{reldg}\{H(z)\}$ is the relative degree of $H(z)$, $\{c_i\}_{i=1,\cdots n_c}$ (resp. $\{p_j\}_{j=1,\cdots n_p}$) is the set of NMP zeros (resp. unstable poles) of $H(z)$, and $\bar{(\cdot)}$ denotes complex conjugation.

All signals in this paper are assumed to be wide sense stationary (wss.) stochastic processes with zero mean and rational power spectral density (PSD). Given a process x, we define its variance as σ_x^2;

$$\sigma_x^2 \triangleq \mathcal{E}\left\{x(k)^2\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\Omega_x(e^{j\omega})\right|^2 d\omega = ||\Omega_x(z)||_2^2 ,$$

where $\Omega_x(z) \in \mathcal{R}$ is such that $\left|\Omega_x(e^{j\omega})\right|^2$ is the PSD of x.

III. Problem Definition

Standard control theory (see, e.g, [1], [2]) assumes that the communication link between sensors and controller, as well as between controller and actuators, is transparent. This situation is depicted in Figure 1, and will be referred to as the *nominal loop*. In Figure 1, $G(z) \in \mathcal{R}$ is the plant transfer function, $C(z) \in \mathcal{R}$ is the controller transfer function, r is the reference signal, y is the plant output (sensor output), u_c is the controller output and $u = u_c$ is the plant input (actuator input). In this architecture, the tracking error

$$e \triangleq r - y \quad (5)$$

satisfies

$$e = S_o(z)r, \quad (6)$$

where

$$S_o(z) \triangleq (1 + G(z)C(z))^{-1} \quad (7)$$

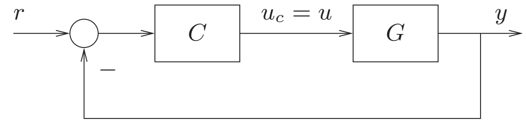

Fig. 1. Standard non-networked control loop.

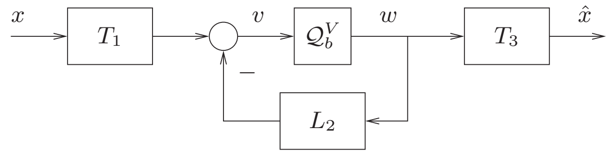

Fig. 2. Noise shaping quantizer.

is the loop sensitivity [1]. The complementary sensitivity is $T_o(z) \triangleq 1 - S_o(z)$.

The new ingredient in NCS's, when compared to the idealized architecture in Figure 1, is that the communication links are not (or cannot be regarded as) ideal. In this setting, quantization, time delays and data drops may have a significative impact on loop performance [7], [8].

In this paper we are interested in the control of the SISO plant $G(z)$ over a bit-rate limited digital communication channel, without data drops or time delays. Thus, the data sent through the channel must be quantized prior to transmission.

A. Quantization scheme and corresponding model

Following standard quantization methods, we will consider a noise shaping quantizer, as depicted in Figure 2 (see also [21], [22]). In Figure 2, x denotes the signal to be quantized, w is a quantized signal that *codes* x in an appropriate fashion, and $\hat{x}$ is an approximation to x. $T_1(z)$, $L_2(z)$ and $T_3(z)$ are scalar proper transfer functions to be designed[1]. $\mathcal{Q}_b^V$ is a $b-$bit uniform quantizer with range $[-V, V]$, i.e.,

$$w(k) = \mathcal{Q}_b^V(v(k)), \forall k \in \mathbb{N}_0, \quad (8)$$

where

$$\mathcal{Q}_b^V(x) = \begin{cases} \left(\left\lfloor \frac{x}{\Delta} \right\rfloor + \frac{1}{2}\right)\Delta & \text{if } -\left(2^{b-1}-1\right)\Delta \le x < \left(2^{b-1}-1\right)\Delta \\ V & \text{if } x \ge \left(2^{b-1}-1\right)\Delta \\ -V & \text{if } x < -\left(2^{b-1}-1\right)\Delta \end{cases} \quad (9)$$

and $\Delta = \frac{2V}{2^b - 1}$.

For the purpose of analysis and design, we will assume that V is such that the probability of quantizer overflow, i.e., $P(|v| > V)$, is negligible, that b is *large* enough, and that v is *random* enough (see discussion in [23], [30], [31]). Under these conditions, it can be assumed that

$$w = v + q, \quad (10)$$

where q is an *independent white noise sequence* uniformly distributed in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$; q denotes *quantization noise*.

[1]Usually, $T_3(z)$ is referred to as an interpolation or reconstruction filter [21]

It is easily shown that the variance of q is given by

$$\sigma_q^2 = \frac{\Delta^2}{12} = \frac{V^2}{3(2^b-1)^2}. \quad (11)$$

On the other hand, it should be noted that, in order to guarantee negligible overflow, V should be selected according to the distribution of v. Therefore, q turns out to be dependent on the quantizer input[2]. In particular, it can be shown that

$$\text{SNR} \triangleq \frac{\sigma_v^2}{\sigma_q^2} \quad (12)$$

can be *fixed* for a given distribution of v by means of an appropriate choice for V [22]. As an illustration, assume that v is a sequence of i.i.d. zero mean Gaussian random variables with variance σ_v^2 and choose $V = 4\sigma_v$. This implies a probability of quantizer overflow of $P(|v| > 4\sigma_v) = 6.33 \cdot 10^{-5}$, which is usually considered to be negligible for practical purposes. This choice for V leads to

$$\text{SNR} = \frac{3}{16} \cdot (2^b-1)^2. \quad (13)$$

B. NCS architecture

The NCS architecture that we will consider in the remainder of this paper is depicted in Figure 3. The key difference between the architecture in Figure 3 and the scheme in Figure 1 is that a communication link has been placed between the controller and plant. In particular, the channel input is encoded with a noise shaping quantizer as described in the last section. The link between the output of the quantizer and the input of $T_3(z)$ is formed by a bit-rate limited channel. We will further assume that no transmission errors occur and that there are no channel delays. Thus, we will assume that $w = \hat{w}$.

Following standard communication nomenclature, we reinterpret the left part of the noise shaping quantizer as an *encoder* and the right part, i.e., $T_3(z)$, as a *decoder*.

The architecture described above enriches the simple pre- and post-filtered PCM quantization-based NCS schemes studied in [25], [27]. Those schemes can be recovered by setting $L_2(z) = 0$. Later we will show that the architecture in Figure 3, in general, attains better performance than the architectures studied in [25], [27].

In order to not alter the nominal relations in (6), we will make the following assumption regarding $T_1(z)$, $T_3(z)$ and $L_2(z)$:

***Assumption** 1:* $T_1(z)$, $T_3(z)$ and $L_2(z)$ are such that

$$T_1(z)\left(1 + L_2(z)\right)^{-1} T_3(z) = 1. \quad (14)$$

We note that the last assumption is equivalent to ensuring that the transfer function between the controller output, u_c, and the plant input, u, is the identity, when no quantization is present. For subsequent reference, we also define

$$S_2(z) \triangleq \left(1 + L_2(z)\right)^{-1} \quad (15)$$

as the sensitivity of the inner loop.

[2] This is, of course, no surprise since quantization is a purely deterministic process and, as a consequence, q depends in a very specific way on v.

In the sequel we will propose an optimal procedure to choose $T_1(z)$, $L_2(z)$ and $T_3(z)$. To that end, we will assume that a satisfactory nominal design has been carried out. In particular, we introduce:

***Assumption** 2:* $C(z)$ is given and stabilizes the nominal loop.

***Remark** 1:* We note that, if Assumptions 1 and 2 hold, and q is assumed to be fully exogenous, then the NCS in Figure 3 is stable (and well possed) if and only if $T_3(z)$ and $S_2(z)$ are stable, minimum phase (MP) and biproper. This follows from standard ideas of avoiding unstable pole-zero cancellations (see, e.g., [1]). ▲

IV. Optimal noise shaping

This section analyzes the NCS architecture proposed in the previous section and describes a procedure to choose $T_1(z)$, $L_2(z)$ and $T_3(z)$.

A. Analysis

From Figure 3 it is straightforward to see that

$$e = r - y = -S_o(z)G(z)T_3(z)S_2(z)q + S_o(z)r, \quad (16)$$

Therefore, the variance of the component of the tracking error due to quantization noise, e_q, is given by

$$\sigma_{e_q}^2 = \sigma_q^2 \left|\left|S_o(z)G(z)T_3(z)S_2(z)\right|\right|_2^2. \quad (17)$$

According to the quantization model in Section III-A, σ_q^2 is not independent. Indeed, using (12) it follows that

$$\sigma_{e_q}^2 = \frac{\sigma_v^2}{\text{SNR}} \left|\left|S_o(z)G(z)T_3(z)S_2(z)\right|\right|_2^2. \quad (18)$$

On the other hand, it is easy to see that (recall (14))

$$v = T_3(z)^{-1}C(z)S_o(z)r - \left(T_3(z)^{-1}C(z)S_o(z)G(z)T_1(z)^{-1} + T_2(z)\right)q, \quad (19)$$

where $T_2(z) \triangleq 1 - S_2(z)$. Therefore,

$$\sigma_v^2 = \left|\left|T_3(z)^{-1}C(z)S_o(z)\Omega_r(z)\right|\right|_2^2 + \sigma_q^2 \left|\left|T_3(z)^{-1}C(z)S_o(z)G(z)T_1(z)^{-1} + T_2(z)\right|\right|_2^2, \quad (20)$$

where we have used the fact that q is independent and has zero mean, and $\left|\Omega_r(e^{j\omega})\right|^2$ is the PSD of the process r. Using (12) and (20) in (18) it follows that

$$\sigma_{e_q}^2 = \frac{\left|\left|T_3(z)^{-1}C(z)S_o(z)\Omega_r(z)\right|\right|_2^2 \left|\left|S_o(z)G(z)T_3(z)S_2(z)\right|\right|_2^2}{\text{SNR} - \left|\left|1 - S_2(z)S_o(z)\right|\right|_2^2}. \quad (21)$$

We note that if $L_2(z) = 0 \Leftrightarrow S_2(z) = 1$, then (21) reduces to

$$\sigma_{e_q}^2 = \frac{\left|\left|T_3(z)^{-1}C(z)S_o(z)\Omega_r(z)\right|\right|_2^2 \left|\left|S_o(z)G(z)T_3(z)\right|\right|_2^2}{\text{SNR} - \left|\left|T_o(z)\right|\right|_2^2}, \quad (22)$$

which is the variance of the component of the tracking error due to quantization noise in the case of pre- and post-filtered PCM quantization, as studied previously in [25], [27].

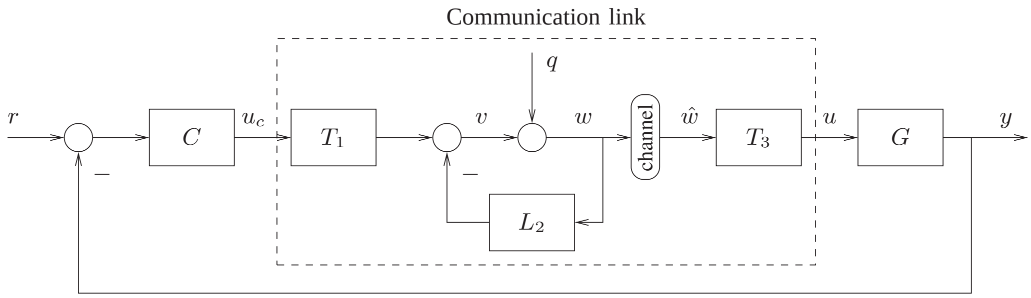

Fig. 3. Noise Shaping NCS Architecture.

B. An analytical solution

This section presents analytical expressions for the transfer functions $T_1(z)$, $L_2(z)$ and $T_3(z)$ which minimize the impact of the communication link on closed loop performance. We do this by introducing a simplifying assumption.

1) Preliminaries: We first present some technical preliminaries.

Define the set

$$\mathcal{S} \triangleq \{H(z) \in \mathcal{R} : H(z) = 1 - M(z), M(z) \in \mathcal{RH}_2\}, \tag{23}$$

and the functional

$$J \triangleq ||S(z)A(z)||_2^2. \tag{24}$$

Also define

$$J_{opt} \triangleq \min_{S(z)\in\mathcal{S}} J, \tag{25}$$

$$S_{opt}(z) \triangleq \arg \min_{S(z)\in\mathcal{S}} J. \tag{26}$$

We then have the following result:

***Lemma** 1:* If $A(z) \in \mathcal{R}$ does not have poles or zeros on the unit circle, then:

1) The optimal $S(z)$ in (26) is given by

$$S_{opt}(z) = \left(\frac{\xi_A(z)A(z)}{\{\xi_A(z)A(z)\}|_{z=\infty}}\right)^{-1}, \tag{27}$$

where $\xi_A(z)$ is a generalized Blaschke product for $A(z)$.

2) The optimal functional value in (25) is given by

$$\begin{aligned} J_{opt} &= (\{\xi_A(z)A(z)\}|_{z=\infty})^2 \\ &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|A(e^{j\omega})\right|^2 d\omega\right). \end{aligned} \tag{28}$$

3) The optimal parameter $S_{opt}(z)$ is such that the spectrum of $S_{opt}(z)A(z)$ is white (i.e., has a constant magnitude for all ω).

Proof:

1) Using the properties of the $2-$norm, the fact that $\left|\xi_A(e^{j\omega})\right| = 1$, and the definition of $\mathcal{S}$, it follows that

$$J = f\left|\left|\left(1 - \frac{1}{z}Q(z)\right)F(z)\right|\right|_2^2, \tag{29}$$

where $Q(z) \in \mathcal{RH}_\infty$ and

$$f \triangleq (\{\xi_A(z)A(z)\}|_{z=\infty})^2, \tag{30}$$

$$F(z) \triangleq \frac{\xi_A(z)A(z)}{\{\xi_A(z)A(z)\}|_{z=\infty}}. \tag{31}$$

Note that, given the definition of $\xi_A(z)$, $F(z)$ is biproper, stable, MP and such that $F(\infty) = 1$. Therefore,

$$\begin{aligned} J &= f\,||zF(z) - Q(z)F(z)|| \\ &= f\,||zF(\infty)||_2^2 + f\,||z(F(z) - F(\infty)) - Q(z)F(z)||_2^2 \\ &= f + f\,||z(F(z) - 1) - Q(z)F(z)||_2^2. \end{aligned} \tag{32}$$

Therefore, the parameter $Q_{opt}(z) \in \mathcal{RH}_\infty$ that minimizes J is given by

$$\begin{aligned} &Q_{opt}(z) = zF(z)^{-1}(F(z) - 1) \\ \Rightarrow\quad &S_{opt}(z) = 1 - \frac{1}{z}Q_{opt}(z) = F(z)^{-1}. \end{aligned} \tag{33}$$

Recalling the definition of $F(z)$, the result (27) follows.

2) The first part of the result follows directly from (32) and the definitions of f and $Q_{opt}(z)$.
To prove that J_{opt} can be written as in (28), it suffices to note that, given the properties of $\xi_A(z)$,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|A(e^{j\omega})\right|^2 d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f\left|F(e^{j\omega})\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f d\omega + \\ &\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|F(e^{j\omega})\right|^2 d\omega. \end{aligned} \tag{34}$$

Now, recall that $F(z)$ is biproper, stable, MP and such that $F(\infty) = 1$. Therefore, it can be regarded as the sensitivity of a loop in which the open loop transfer function is *stable and strictly proper*. Using the Bode integral (see, e.g., [1], [32]) it follows that the last integral in (34) equals zero. Therefore,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|A(e^{j\omega})\right|^2 d\omega = \ln f. \tag{35}$$

The result follows.

3) Immediate form part (1) and the properties of $\xi_A(z)$.

■

Remark 2: If $A(z)$ does have poles and/or zeros on the unit circle, then it can be proved, using a procedure similar to that in [33] (Lemma 10, p. 171), that J_{opt} (defined by an infimum and not a minimum as above) is still as in Lemma 1. Then, one can construct a sequence of stable $S(z)$'s that achieve a cost J as close to J_{opt} as desired. Note, however, that there is no stable $S(z)$ that achieves J_{opt}. ▲

Remark 3: Assume a linear system with transfer function $S(z) \in \mathcal{S}$ whose input is an arbitrary wide sense stationary stochastic process x with PSD

$$\left|\Omega_x(e^{j\omega})\right|^2 = \left|H_x(e^{j\omega})\right|^2 \sigma_{u_x}^2,$$

where $H_x(z)$ is such that $H_x(\infty) = 1$ and having no poles or zeros on the unit circle. $\sigma_{u_x}^2$ is the variance of the white noise sequence u_x that excites $H_x(z)$ to generate x.

Under the conditions of the last paragraph, Lemma 1 implies that the transfer function $S(z) \in \mathcal{S}$ that achieves *minimum variance* at its output when its input is x, is such that its output is white noise, with the same variance as the variance of the white noise considered when generating x. We note that these conclusions are consistent with minimum variance control for MP plants [34]. ▲

Based on the previous observations we see that Lemma 1 shows that the optimal parameter $S(z) \in \mathcal{S}$ is such that it *whitens* the spectrum of $A(z)$. Accordingly, we will use the term *optimal whitening filter for* $A(z)$ to refer to $S_{opt}(z)$.

2) Simplifying assumption: We will next apply the results in Lemma 1 to derive optimal values of $T_1(z)$, $L_2(z)$ and $T_3(z)$ in Figure 3. To achieve a closed form solution, we make the following simplifying assumption:

$$\text{SNR} - \|1 - S_2(z)S_o(z)\|_2^2 \approx \text{SNR}, \tag{36}$$

which implies (see (21))

$$\sigma_{e_q}^2 \approx \hat{J} \triangleq \frac{\left\|T_3(z)^{-1}C(z)S_o(z)\Omega_r(z)\right\|_2^2 \|S_o(z)G(z)T_3(z)S_2(z)\|_2^2}{\text{SNR}}. \tag{37}$$

We note that this restriction is actually quite mild. Indeed, we will see in Section V that the consequences of this assumption lead to valid performance predictions even for very low bit-rates. Clearly, $\hat{J}$ is the (approximate) variance of the component of the tracking error due to quantization noise.

As is common practice in actual implementations of noise shaping quantizers (see, e.g., [22]), we will also assume the following:

Assumption 3: $T_3(z) \in \mathcal{S}$ (see definition in (23)).

Assumptions 1, 2 and 3 (recall Remark 1) imply that we are interested in the set of triplets $(T_1(z), L_2(z), T_3(z))$ defined by

$$\mathcal{O} \triangleq \{(T_1(z), L_2(z), T_3(z)) \in \mathcal{R}^3 : T_3(z) \in \mathcal{S} \text{ and is MP}, \; (1 + L_2(z))^{-1} \text{ is stable}, \; \text{MP and biproper, and } T_1(z)\,(1 + L_2(z))^{-1}\,T_3(z) = 1.\} \tag{38}$$

Accordingly, we define

$$\hat{J}_{opt} \triangleq \min_{(T_1(z), L_2(z), T_3(z)) \in \mathcal{O}} \hat{J} \tag{39}$$

and

$$(\hat{T}_{1\,opt}(z), \hat{L}_{2\,opt}(z), \hat{T}_{3\,opt}(z)) \triangleq \arg \min_{(T_1(z), L_2(z), T_3(z)) \in \mathcal{O}} \hat{J}. \tag{40}$$

3) Key Result: We are now in a position to state the key result of this paper:

Theorem 1: Provided $C(z)S(z)\Omega_r(z)$ and $S_o(z)G(z)$ have no poles or zeros on the unit circle, then:

1) The optimal noise shaping parameters defined in (40) are given by

$$\hat{T}_{1\,opt}(z) = \frac{\xi_2(z)S_o(z)G(z)}{\{\xi_2(z)S_o(z)G(z)\}|_{z=\infty}} \tag{41}$$

$$\hat{T}_{3\,opt}(z) = \frac{\xi_1(z)C(z)S_o(z)\Omega_r(z)}{\{\xi_1(z)C(z)S_o(z)\Omega_r(z)\}|_{z=\infty}} \tag{42}$$

$$\hat{L}_{2\,opt}(z) = \hat{T}_{3\,opt}(z)\hat{T}_{1\,opt}(z) - 1, \tag{43}$$

where $\xi_1(z), \xi_2(z)$ are generalized Blaschke products for $C(z)S_o(z)\Omega_r(z)$ and $S_o(z)G(z)$, respectively.

2) The corresponding optimal loss function is given by

$$\hat{J}_{opt} = \frac{1}{SNR}\left(\{\xi_1(z)\xi_2(z)T_o(z)S_o(z)\Omega_r(z)\}|_{z=\infty}\right)^2. \tag{44}$$

Proof:

1) We write

$$\hat{J} = \frac{1}{SNR} J_1 J_2, \tag{45}$$

where

$$J_1 = \left\|T_3(z)^{-1}C(z)S_o(z)\Omega_r(z)\right\|_2^2,$$
$$J_2 = \|S_o(z)G(z)T_3(z)S_2(z)\|_2^2. \tag{46}$$

Since there are two degrees of freedom available, we can minimize $\hat{J}$ in a two step procedure. We first minimize J_1 by choosing $T_3(z)$ and we then minimize J_2 by choosing $S_2(z)$.

Since $T_3(z) \in \mathcal{S}$ and is MP, it follows that $T_3(z)^{-1} \in \mathcal{S}$. In addition, $C(z)S_o(z)\Omega_r(z)$ is assumed to have no poles or zeros on the unit circle. Therefore we can immediately utilize Lemma 1 giving

$$\hat{T}_{3\,opt}(z) = \frac{\xi_1(z)C(z)S_o(z)\Omega_r(z)}{\{\xi_1(z)C(z)S_o(z)\Omega_r(z)\}|_{z=\infty}}. \tag{47}$$

Substituting $\hat{T}_{3\,opt}(z)$ into J_2 yields

$$J_2 = \left\|S_o(z)G(z)\hat{T}_{3\,opt}(z)S_2(z)\right\|_2^2. \tag{48}$$

We note that, in order for the inner loop in Figure 3 to be well defined, $L_2(z)$ must be strictly proper, i.e., there must exist a proper $\bar{C}(z)$ such that $L_2(z) = z^{-1}\bar{C}(z)$. Therefore, $S_2(z)$ is an admissible sensitivity function (i.e., a sensitivity function originating in a stabilizing and proper controller $\bar{C}(z)$) for the plant z^{-1}. The

Youla parameterization (see, e.g., [1]) ensures that all admissible $S_2(z)$ can be written as

$$S_2(z) = 1 - \frac{1}{z}Q(z), \tag{49}$$

where $Q(z) \in \mathcal{RH}_\infty$. As a consequence, $S_2(z) \in \mathcal{S}$. Since, in addition, $S_o(z)G(z)$ is assumed to have no poles or zeros on the unit circle, and $\hat{T}_{3\,opt}(z)$ is stable, MP and biproper, we can use Lemma 1 again to obtain

$$\hat{S}_{2\,opt}(z) = \left(\frac{\xi_2(z)S_o(z)G(z)}{\{\xi_2(z)S_o(z)G(z)\}|_{z=\infty}} \hat{T}_{3\,opt}(z) \right)^{-1}. \tag{50}$$

Therefore,

$$\hat{L}_{2\,opt}(z) = \frac{\xi_2(z)S_o(z)G(z)}{\{\xi_2(z)S_o(z)G(z)\}|_{z=\infty}} \hat{T}_{3\,opt}(z) - 1. \tag{51}$$

Finally, using the fact that $T_1(z)S_2(z)T_3(z) = 1$, it follows that

$$\hat{T}_{1\,opt}(z) = \frac{\xi_2(z)S_o(z)G(z)}{\{\xi_2(z)S_o(z)G(z)\}|_{z=\infty}}. \tag{52}$$

Given the properties of generalized Blaschke products, the optimal triplet defined in the previous paragraphs belongs to $\mathcal{O}$, as required.

2) Immediate form the part 1 and Lemma 1.

■

***Remark** 4:* Theorem 1 can be extended to cases in which $C(z)S(z)\Omega_r(z)$ and/or $S_o(z)G(z)$ have poles or zeros on the unit circle as discussed in Remark 2 ▲.

We also have the following corollary to Theorem 1:

***Corollary** 1:* Under the conditions of Theorem 1, $\hat{L}_{2\,opt}(z) = 0$ if and only if $T_o(z)S_o(z)\Omega_r(z)$ is a constant.

Proof: The result follows from Theorem 1, the definition of loop sensitivities and the fact that, given the properties of Blaschke products, having $\xi_A(z)A(z)$ constant is equivalent to having $A(z)$ constant. ■

The above corollary implies that (save for the very special case in which $T_o(z)S_o(z)\Omega_r(z)$ is constant) the best achievable performance of the novel noise shaping NCS studied in this paper is guaranteed to be better than the performance of the pre- and post-filtered PCM quantization-based NCS schemes studied previously in [25], [27]. As stated before, this is as expected given the fact that the PCM-based scheme is a special case of the noise shaping scheme.

Further insight into the nature of the optimal noise shaping NCS can be gained by noting that, under the assumption that the SNR is large, then q is *small* and hence (19) reduces to

$$v \approx T_3(z)^{-1}C(z)S_o(z)r. \tag{53}$$

Under these conditions, the optimal choice for $T_3(z)$ is such that it *whitens the input to the quantizer*, v. Furthermore, once v has been whitened, the optimal choice for $L_2(z)$ (equivalently $S_2(z)$) is such that it *whitens the effect of the quantization noise on the tracking error* (see (16)).

V. EXAMPLE

This section presents a simple example to illustrate the results in this paper.

A. Nominal design

We consider a continuous time plant given by $G_o(s) = 2(5s+1)^{-1}$, sampled every $T = 1[s]$ using a zero order hold at its input. The corresponding discrete time transfer function is

$$G(z) = \frac{0.36254}{(z - 0.8187)}.$$

We will consider two different reference signals, r_1 and r_2, with PSD's given by

$$\left|\Omega_{r_1}(e^{j\omega})\right|^2 = \left|\frac{0.02}{e^{j\omega}-1}\right|^2,$$

$$\left|\Omega_{r_2}(e^{j\omega})\right|^2 = \left|\frac{0.03}{(e^{j\omega}-0.9)(e^{j\omega}-0.7)}\right|^2.$$

For the control of $G(z)$ we choose the PI controller

$$C(z) = \frac{2.4488(z - 0.4871)}{(z-1)}.$$

This controller is assumed to give satisfactory performance in the absence of channel quantization.

B. The case of r_1

In this case, $T_o(e^{j\omega})S_o(e^{j\omega})\Omega_{r_1}(e^{j\omega})$ is approximately constant for all ω. Then, Corollary 1 suggests that the PCM based scheme described in [25], [27] (called the PCM scheme in the sequel) should have a performance which is close to that of the noise shaping based scheme studied in this paper. This is confirmed by the results in Figure 4. In that figure, we show the *empirical*, i.e., simulated (using an actual uniform quantizer), loop error sample variance as a function of the bit-rate b/T of the communication channel, considering several coding schemes. It is clear that the closed loop performance can be significantly improved through (PCM or noise shaping based) coding. It is worth noting that for high bit-rates, the performance of the networked control loop (with and without coding) is almost identical to the nominal performance. This is a consequence of the associated high signal-to-noise ratio (see (21)).

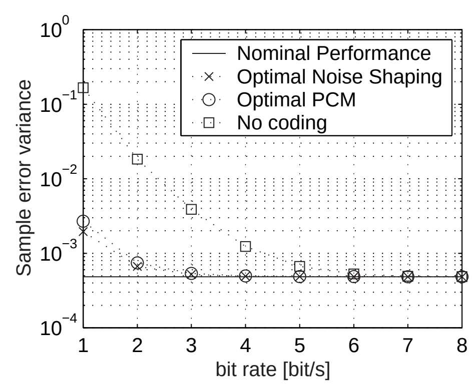

Fig. 4. Sample loop error variance as a function of the channel bit-rate ($r = r_1$).

C. The case of r_2

Figure 5 shows the empirical loop error sample variance when considering r_2 as reference together with various coding schemes. In this case, $T_o(e^{j\omega})S_o(e^{j\omega})\Omega_{r_2}(e^{j\omega})$ is far from being constant. Therefore, the fact that the noise shaping coder system outperforms PCM is no surprise. Again, the benefits of coding are apparent.

D. Testing impact of simplifying assumption

In Figure 6 we show a comparison between the theoretical variance of the loop error as given by Theorem 1 and the empirical loop error sample variance, when $r = r_2$ and optimal noise shaping coding is employed. In Figure 6, $\sigma^2_{e_r}$ refers to the theoretical variance of the effect of the reference on the tracking error, i.e., $||S_o(z)\Omega_r(z)||_2^2$ (see (16)). The *empirical* results refer to the observed variance under simulated conditions. Except for the extreme case $b = 1$, a very close match is obtained, and even for $b = 1$, the qualitative performance is as predicted.

VI. CONCLUSIONS

This paper has studied a novel noise shaping quantizer based coder/decoder system for NCS's in which the communication between controller and plant is performed over a bit-rate limited digital channel. It has been shown how the various design parameters can be chosen so as to minimize the impact of channel imperfections on loop performance. The results have been confirmed by non idealized simulations using a bit-rate limited communication channel.

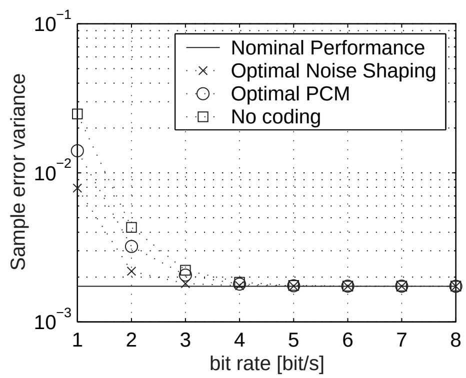

Fig. 5. Sample loop error variance as a function of the channel bit-rate ($r = r_2$).

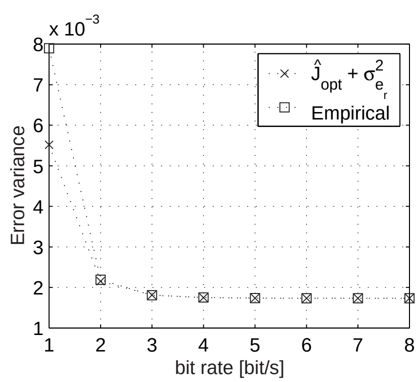

Fig. 6. Comparison between analytical performance index values and empirical values with $r = r_2$ and optimal noise shaping coding.

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