INTRODUCTION TO WAVELETS

Adapted from
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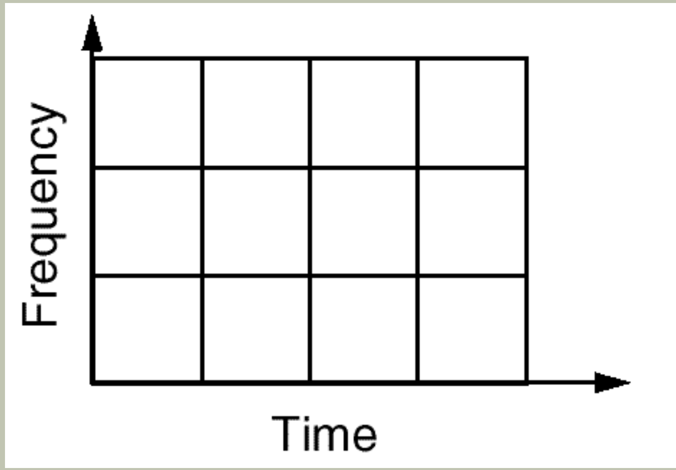
CRITICISM OF FOURIER SPECTRUM

- It gives us the spectrum of the 'whole time-series'
  - Which is OK if the time-series is stationary
  - But what if its not?

- We need a technique that can "march along" a time series and that is capable of:
  - Analyzing spectral content in different places
  - Detecting sharp changes in spectral character

SHORT TIME FOURIER TRANSFORM STFT

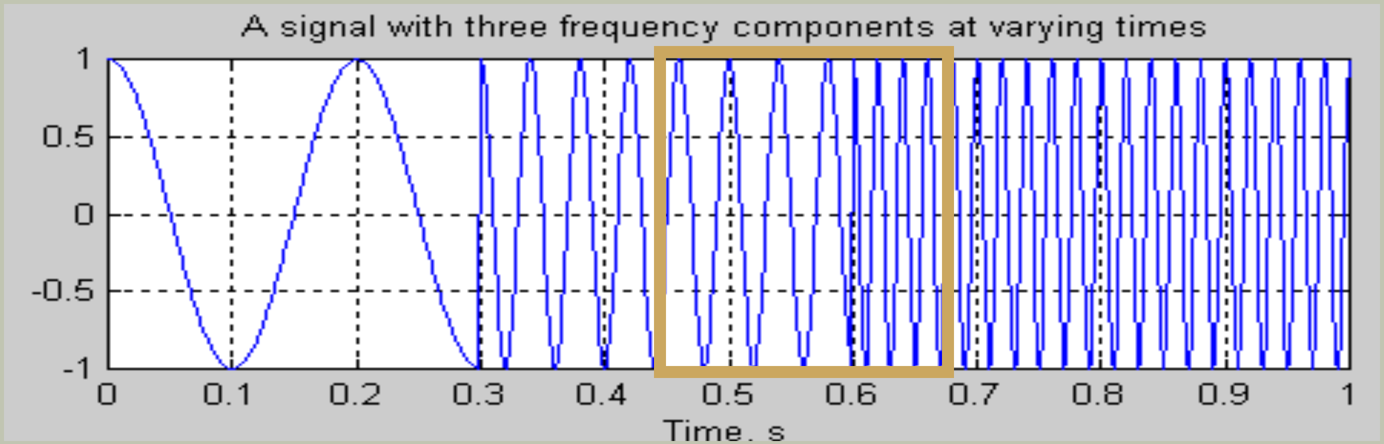
- Time/Frequency localization depends on window size.
- Once you choose a particular window size, it will be the same for all frequencies.
- Many signals require a more flexible approach - vary the window size to determine more accurately either time or frequency.



THE WAVELET TRANSFORM

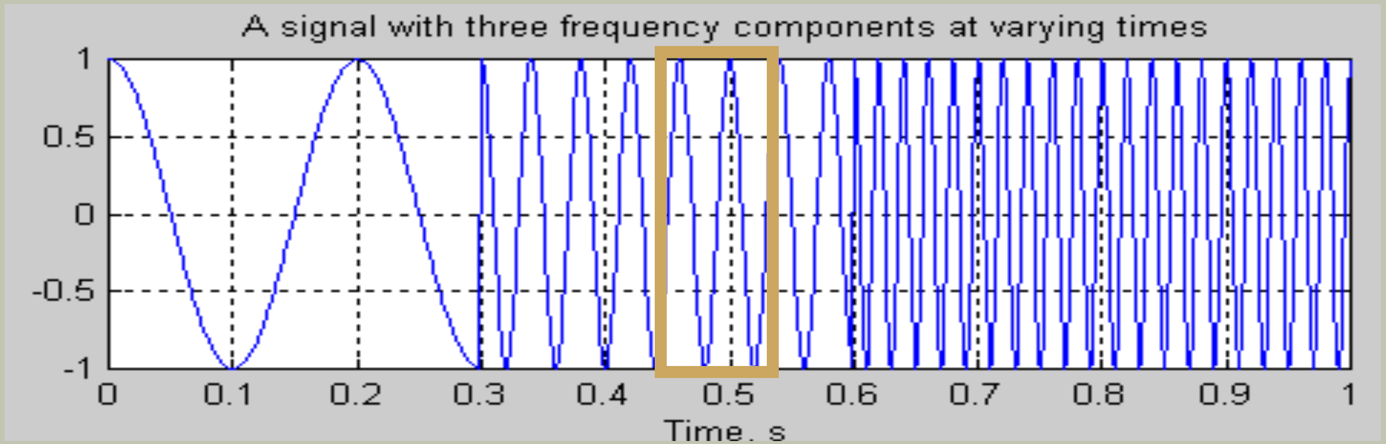
- Overcomes the preset resolution problem of the STFT by using a variable length window:
  - Use narrower windows at high frequencies for better time resolution.
  - Use wider windows at low frequencies for better frequency resolution.

The Wavelet Transform (cont'd)



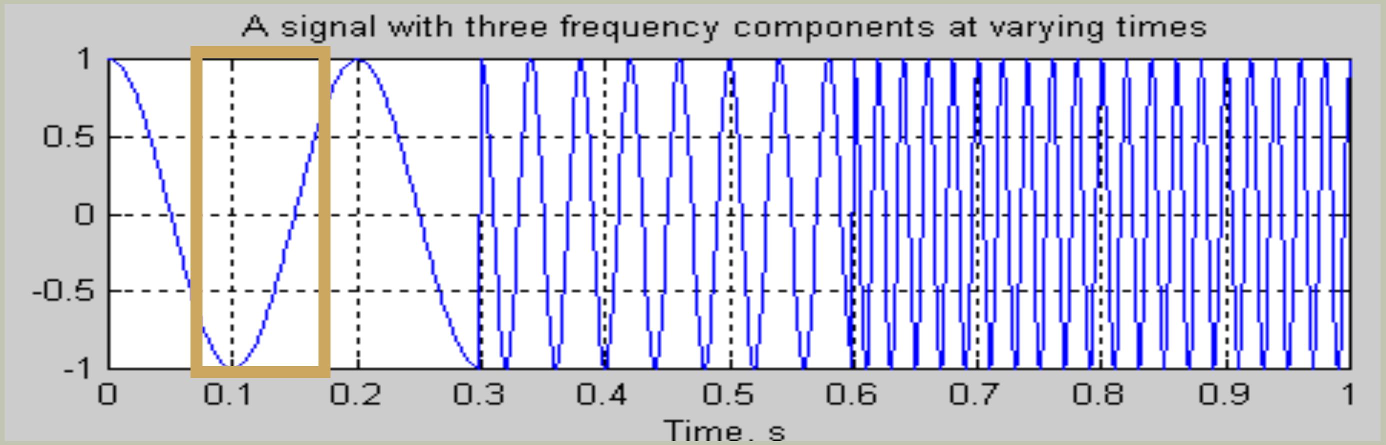
Wide windows do not provide good localization at high frequencies.

The Wavelet Transform (cont'd)



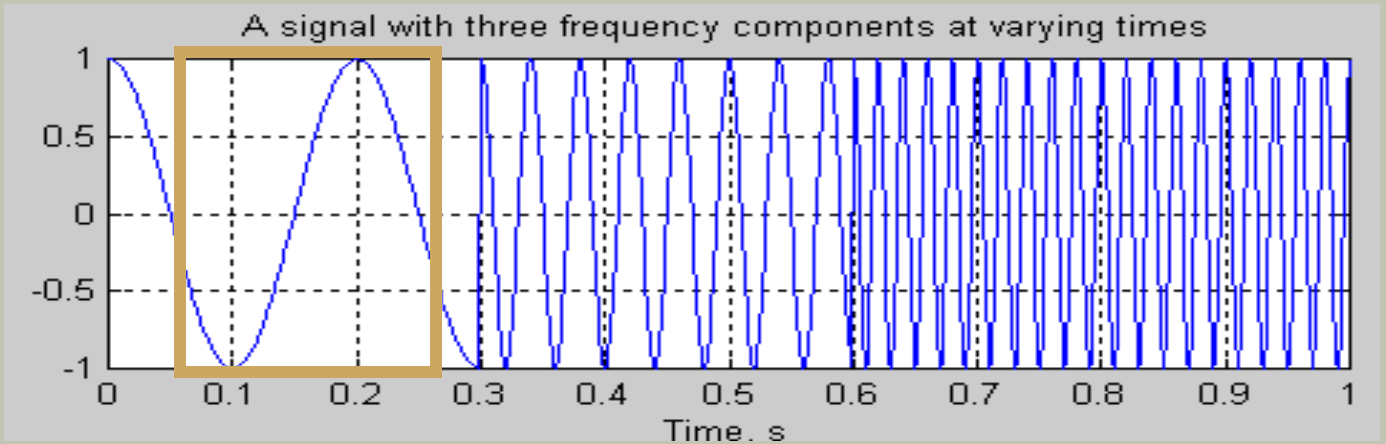
Use narrower windows at high frequencies.

The Wavelet Transform (cont'd)



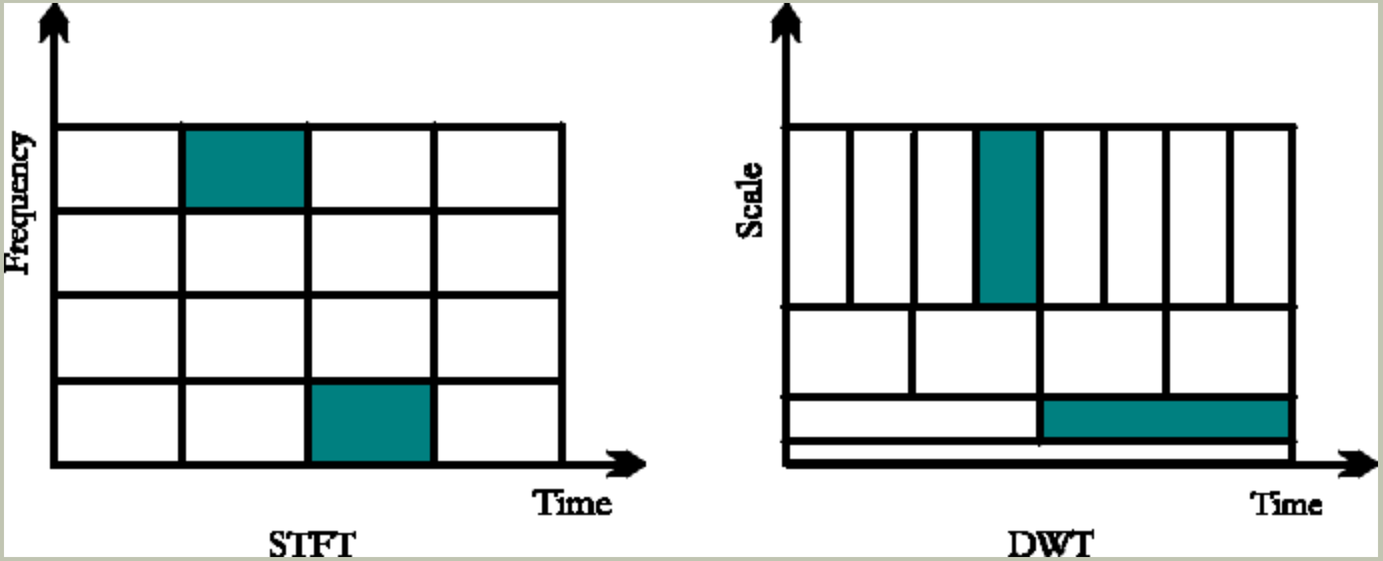
Narrow windows do not provide good localization at low frequencies.

The Wavelet Transform (cont'd)



Use wider windows at low frequencies.

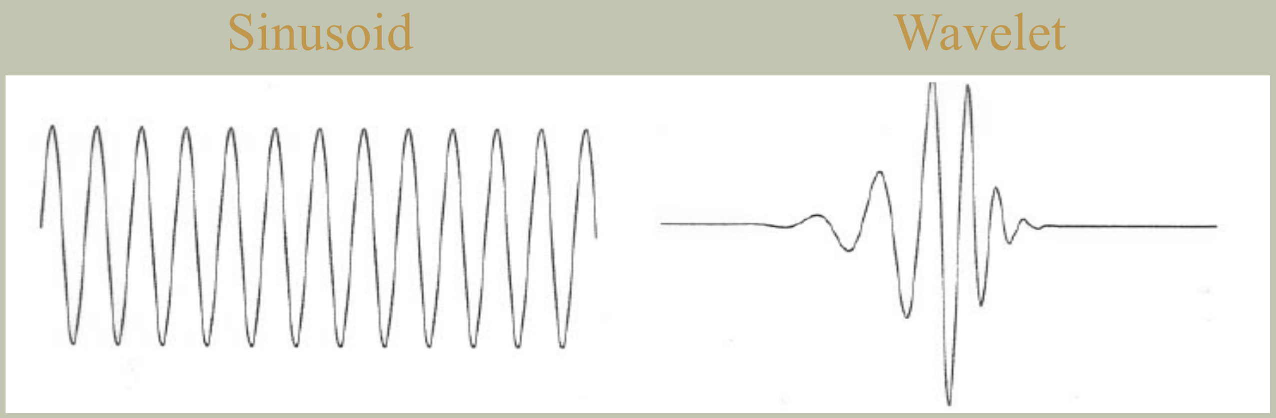
The Wavelet Transform (cont'd)



STFT AND DWT BREAKDOWN OF A SIGNAL

WHAT ARE WAVELETS?

- Wavelets are functions that "wave" above and below the x-axis, have (1) varying frequency, (2) limited duration, and (3) an average value of zero.
- This is in contrast to sinusoids, used by FT, which have infinite energy.



What are Wavelets? (cont'd)

- Like sines and cosines in FT, wavelets are used as basis functions $\psi_k(t)$ in representing other functions f(t):

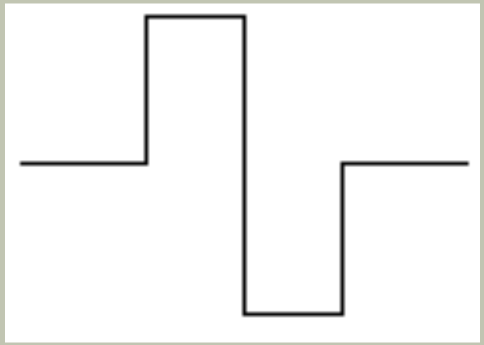
$$f(t) = \sum_k a_k \psi_k(t)$$

- Span of $\psi_k(t)$: vector space S containing all functions f(t) that can be represented by $\psi_k(t)$.

What are Wavelets? (cont'd)

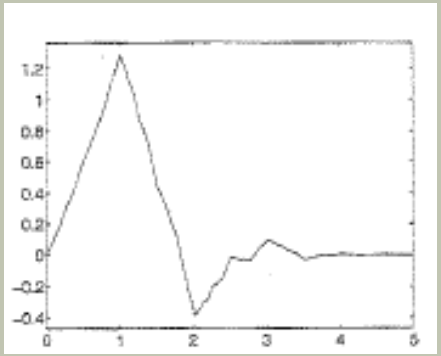
- There are many different wavelets:

Haar



Morlet

Daubechies



What are Wavelets? (cont'd)

- Once the mother wavelet ψ(t) is fixed, one can form a basis from it by applying translations and scalings (i.e., stretch/compress):

$$\psi(s,\tau,t) = \frac{1}{\sqrt{s}}\psi(\frac{t-\tau}{s})$$

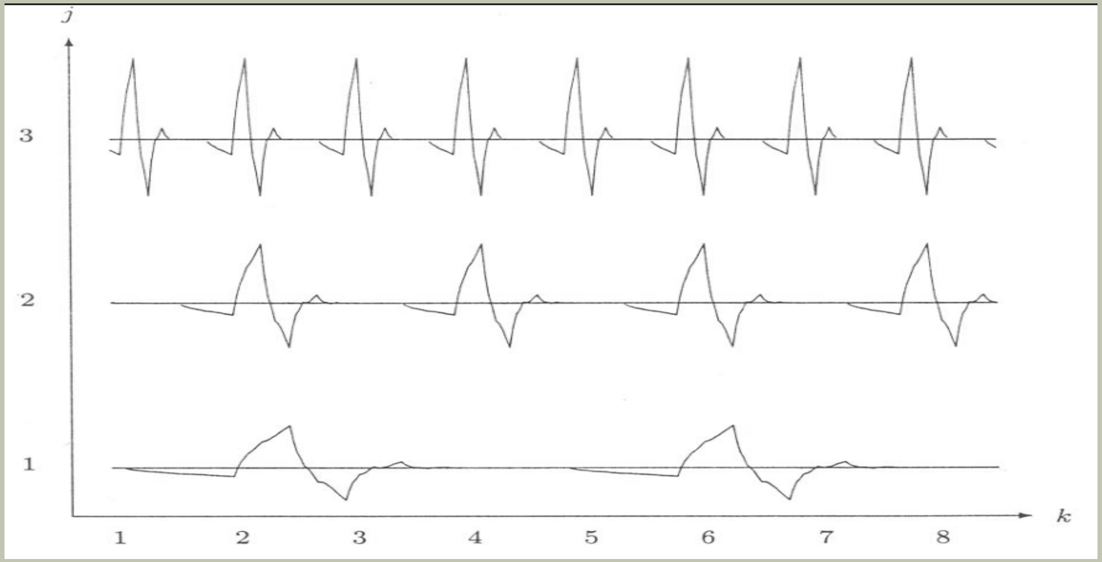
- It is convenient to take special values for s and τ in defining the wavelet basis: **$s = 2^{-j}$** and **$\tau = k \,.\, 2^{-j}$**

$$\psi(s,\tau,t) = \frac{1}{\sqrt{2^{-j}}}\psi\left(\frac{t-k.2^{-j})}{2^{-j}}\right) = 2^{\frac{j}{2}}\psi(2^{j}t-\mathrm{k}) = \psi_{jk}(t)$$

What are Wavelets? (cont'd)

$$\psi_{jk}(t) = 2^{j/2}\psi\left(2^j t - k\right)$$

scale/frequency localization



j

time localization

MORE ABOUT WAVELETS

normalization

shift in time

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

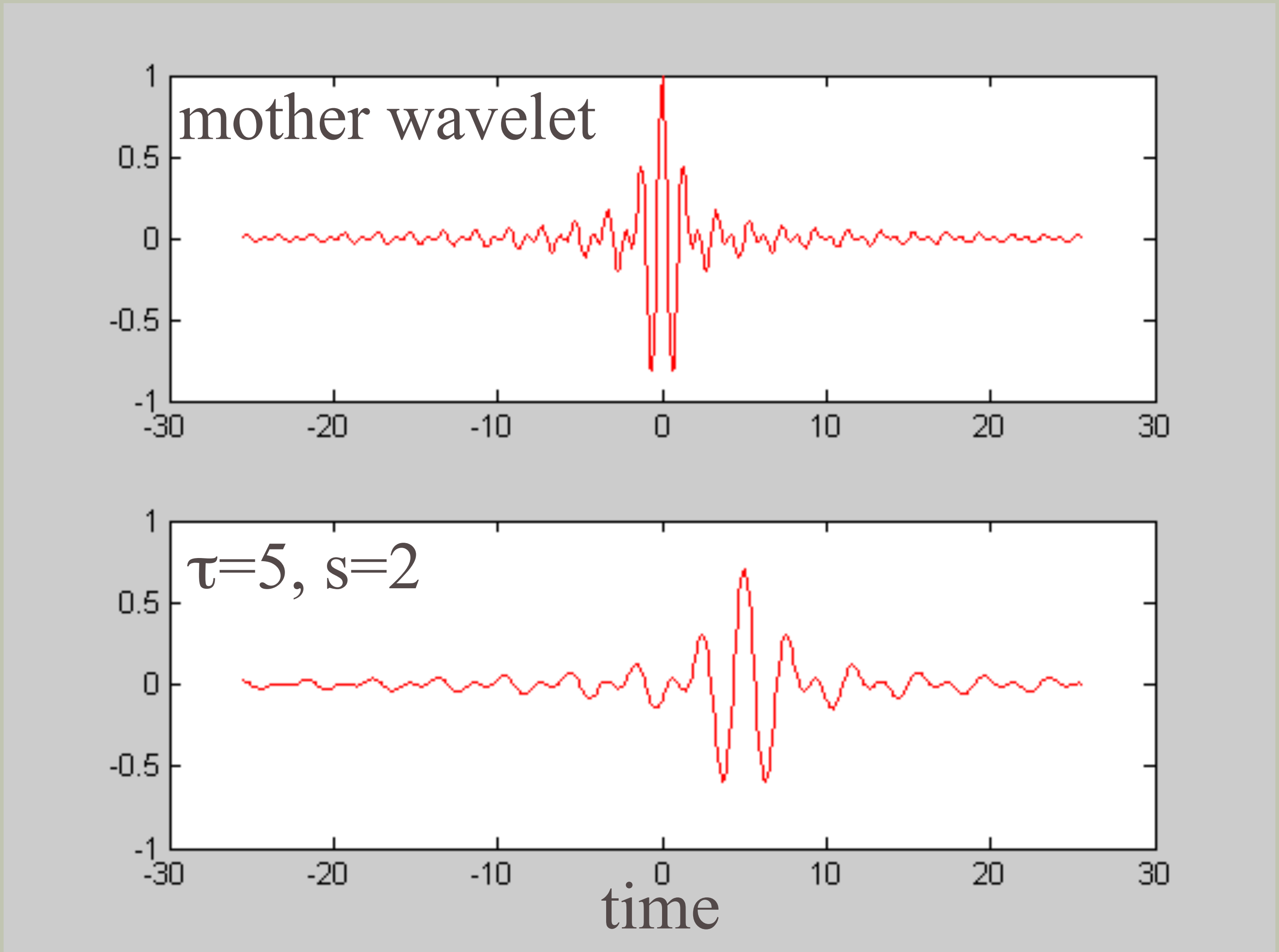
change in scale: big s means long wavelength

wavelet with scale, s and time, τ

Mother wavelet

SHANNON WAVELET

$$Y(t) = 2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)$$



CONTINUOUS WAVELET TRANSFORM (CWT)

translation parameter, measure of time

scale parameter (measure of frequency)

normalization constant

Forward CWT:

$$C(\tau, s) = \frac{1}{\sqrt{s}} \int_t f(t) \psi^* \left(\frac{t - \tau}{s} \right) dt$$

Continuous Wavelet Transform of signal *f(t)*

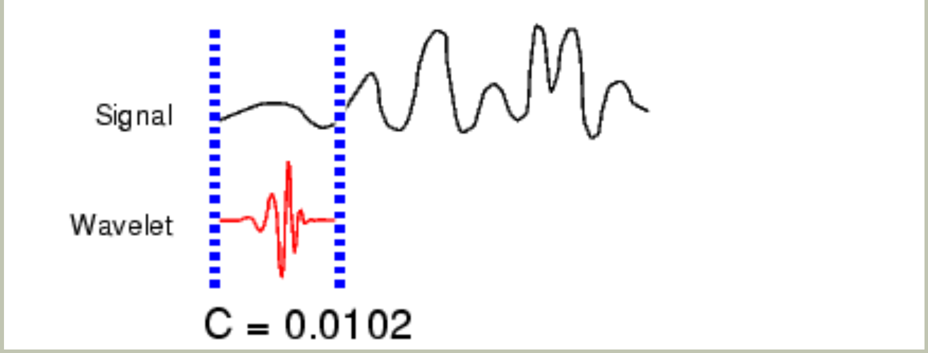
Mother wavelet (window)

Scale = 1/j = 1/Frequency

CWT: MAIN STEPS

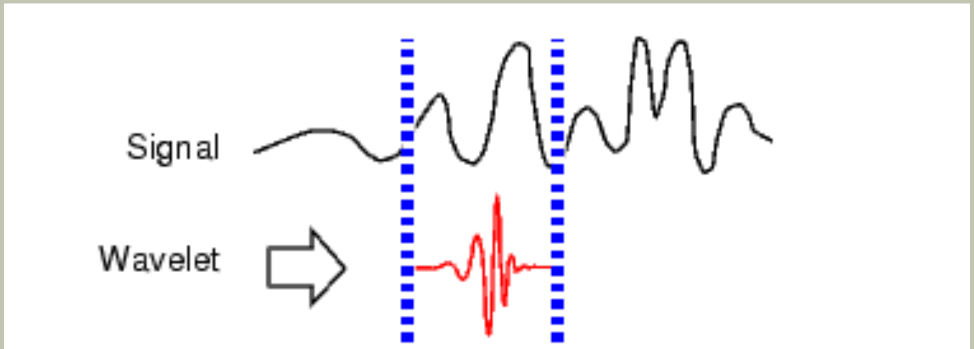
1. Take a wavelet and compare it to a section at the start of the original signal.

2. Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity.



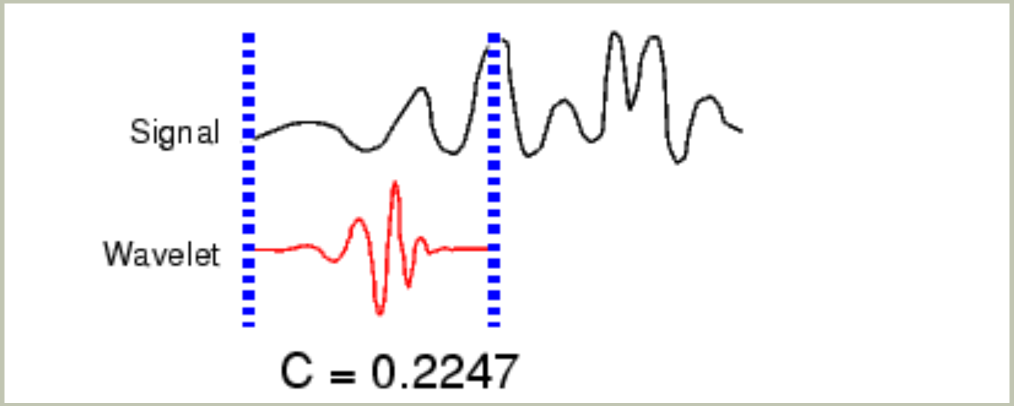
CWT: Main Steps (cont'd)

3. Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.



CWT: Main Steps (cont'd)

4. Scale the wavelet and repeat steps 1 through 3.

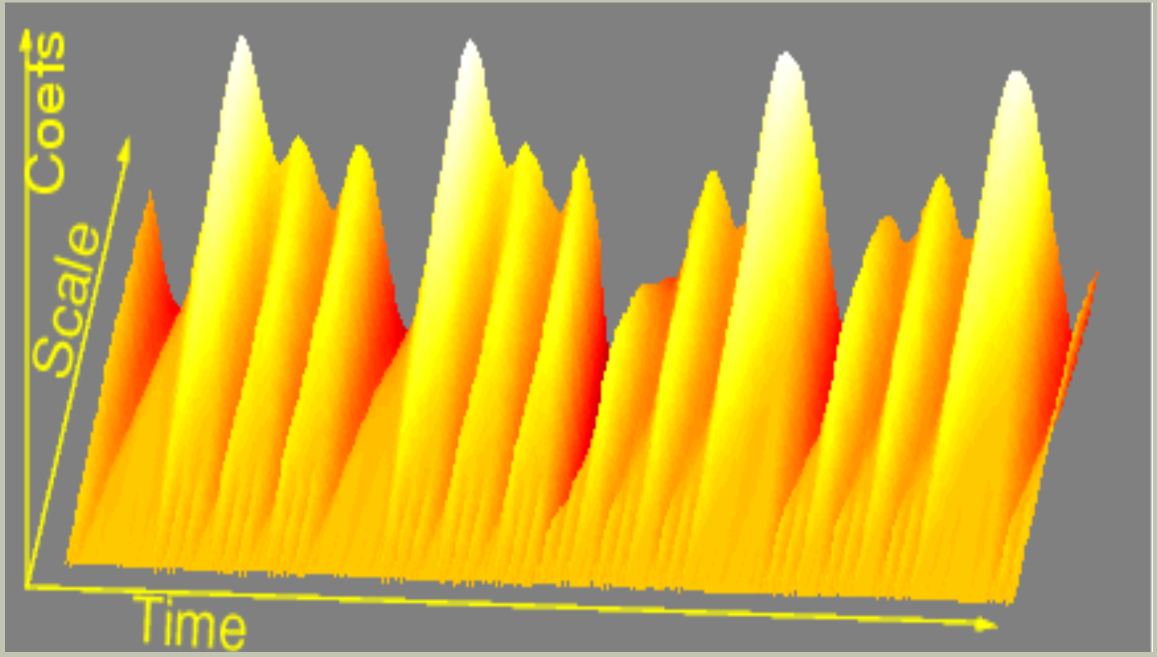


5. Repeat steps 1 through 4 for all scales.

COEFFICIENTS OF CTW TRANSFORM

- Wavelet analysis produces a time-scale view of the input signal or image.

$$C(\tau, s) = \frac{1}{\sqrt{s}} \int_t f(t) \psi^* \left(\frac{t - \tau}{s} \right) dt$$



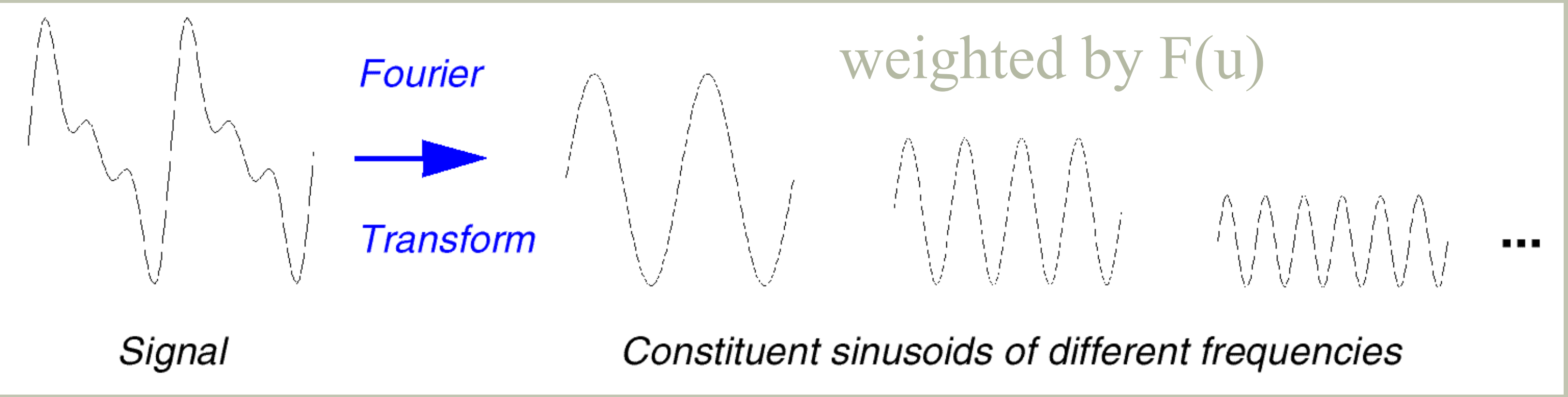
Continuous Wavelet Transform (cont'd)

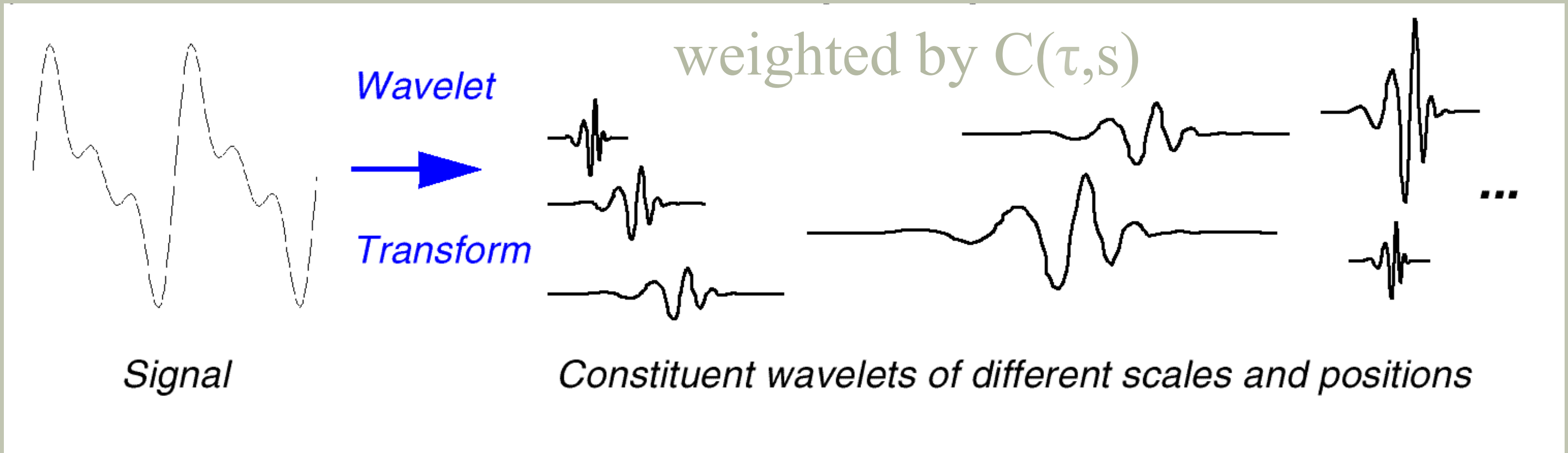
- Inverse CWT:

$$f(t) = \frac{1}{\sqrt{s}} \int_{\tau}\int_{s} C(\tau, s)\psi(\frac{t-\tau}{s}) d\tau ds$$

double integral!

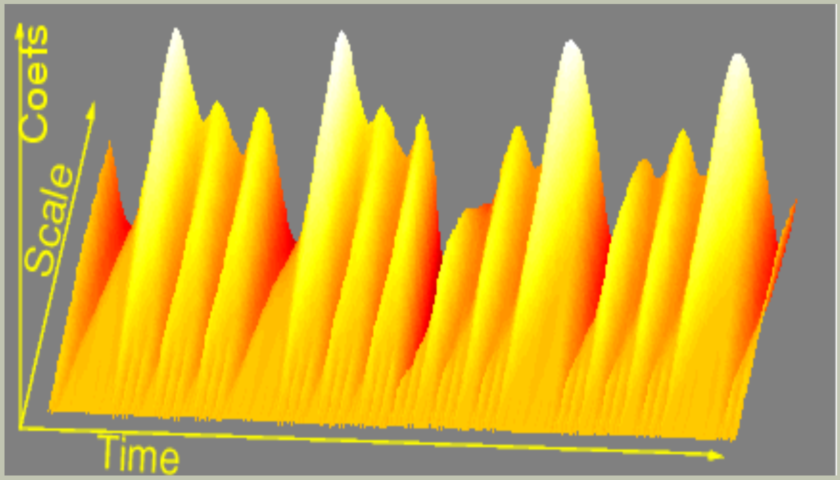
FT VS WT


weighted by F(u)
Fourier
Transform
...
Signal
Constituent sinusoids of different frequencies


weighted by C(τ,s)
Wavelet
Transform
...
Signal
Constituent wavelets of different scales and positions

PROPERTIES OF WAVELETS

- Simultaneous localization in time and scale
  - The location of the wavelet allows to explicitly represent the location of events in time.
  - The shape of the wavelet allows to represent different detail or resolution.



Properties of Wavelets (cont'd)

- Sparsity: for functions typically found in practice, many of the coefficients in a wavelet representation are either zero or very small.

$$f(t) = \frac{1}{\sqrt{s}} \int_{\tau}\int_{s} C(\tau, s)\psi\left(\frac{t-\tau}{s}\right) d\tau ds$$

- Linear-time complexity: many wavelet transformations can be accomplished in O(N) time.

Properties of Wavelets (cont'd)

- Adaptability: wavelets can be adapted to represent a wide variety of functions (e.g., functions with discontinuities, functions defined on bounded domains etc.).
  - Well suited to problems involving images, open or closed curves, and surfaces of just about any variety.
  - Can represent functions with discontinuities or corners more efficiently (i.e., some have sharp corners themselves).

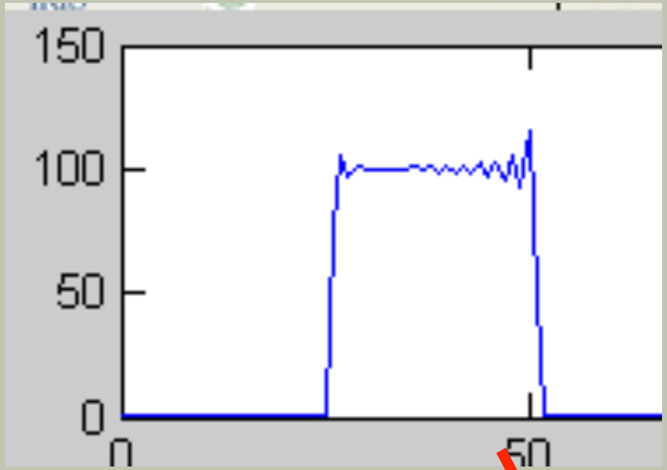
Properties of Wavelets (cont'd)

- Admissibility condition:

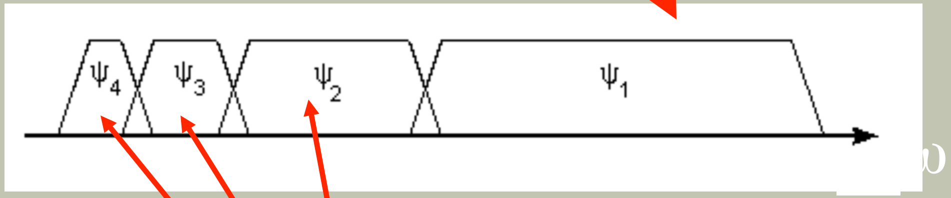
$$\int \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < +\infty$$

Implies that $\Psi(\omega) \rightarrow 0$ both as $\omega \rightarrow 0$ and $\omega \rightarrow \infty$, so $\Psi(\omega)$ must be *band-limited*

Fourier spectrum of Shannon Wavelet



frequency, ω



Spectrum of higher scale wavelets

DISCRETE WAVELET TRANSFORM (DWT)

- CWT computes all scales and positions in a given range
- DWT scales and positions are only computed in powers of 2 (dyadic scales)
- This subset can be shown to have the same accuracy as DWT
- Dyadic scales allow for tree decompositions

DISCRETE WAVELET TRANSFORM (DWT)

$$a_{jk} = \sum_t f(t)\psi^*_{jk}(t)$$ (forward DWT)

$$f(t) = \sum_k \sum_j a_{jk}\psi_{jk}(t)$$ (inverse DWT)

where $$\psi_{jk}(t) = 2^{j/2}\psi\left(2^j t - k\right)$$

DFT VS DWT

- DFT expansion:

one parameter basis

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{\frac{j2\pi ux}{N}},$$

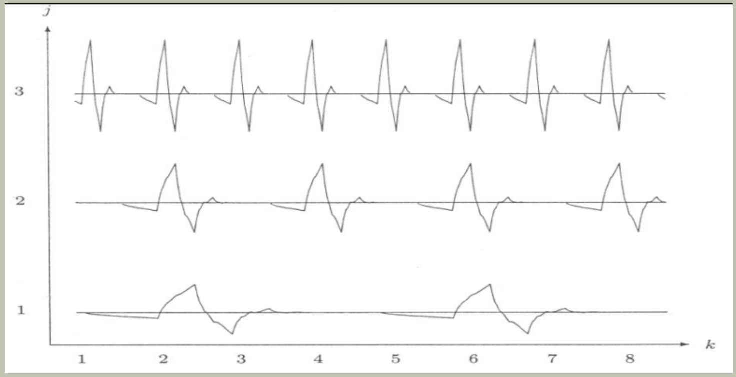
or

$$f(t) = \sum_{l} a_l \psi_l(t)$$

- DWT expansion

two parameter basis

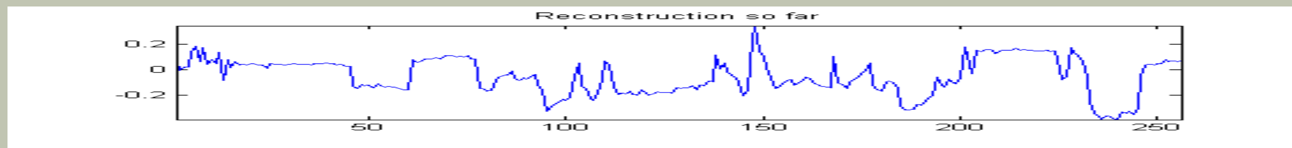
$$f(t) = \sum_{k} \sum_{j} a_{jk} \psi_{jk}(t)$$

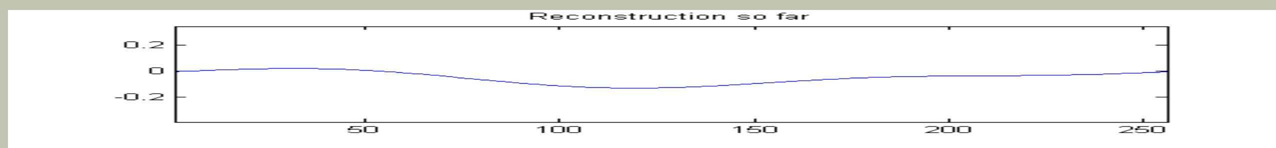


MULTIRESOLUTION REPRESENTATION USING

$\psi_{jk}(t)$

$f(t)$

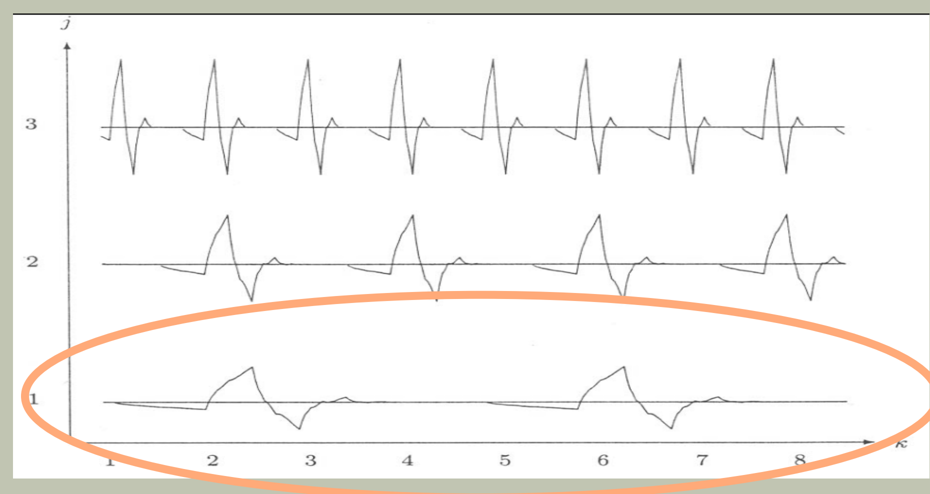




fine details

wider, large translations

$$f(t) = \sum_{k} \sum_{j} a_{jk} \psi_{jk}(t)$$

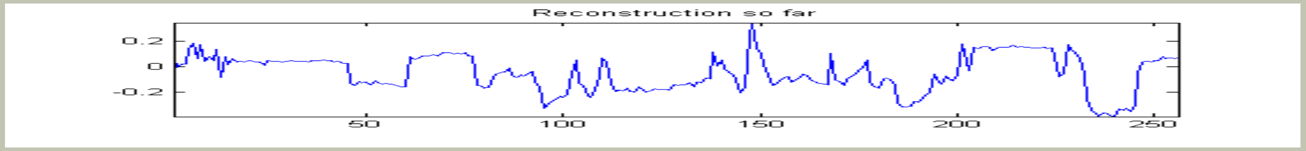


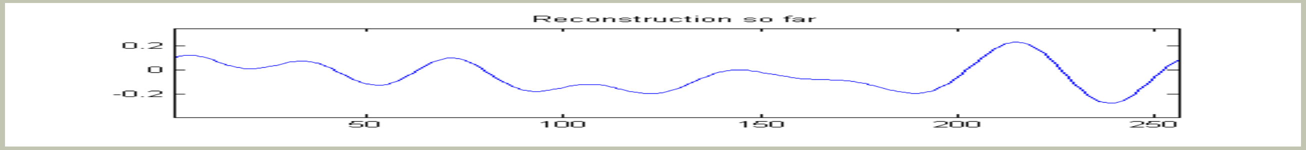
j

coarse details

MULTIRESOLUTION REPRESENTATION USING $\psi_{jk}(t)$

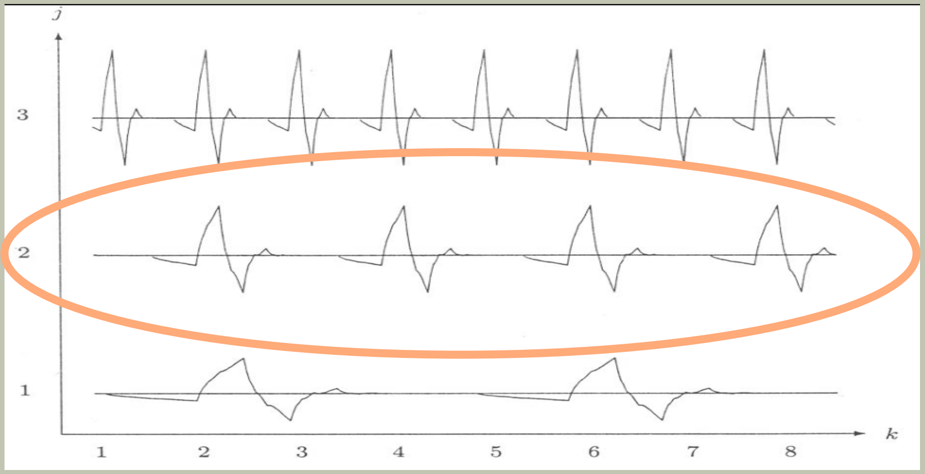
$f(t)$





$$f(t) = \sum_{k} \sum_{j} a_{jk} \psi_{jk}(t)$$

fine details



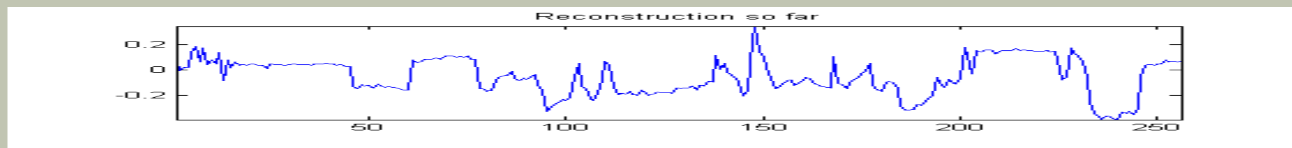
j

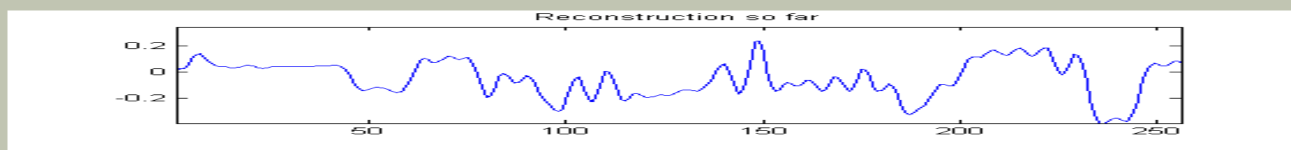
coarse details

MULTIRESOLUTION REPRESENTATION USING

$$\psi_{jk}(t)$$

$$f(t)$$

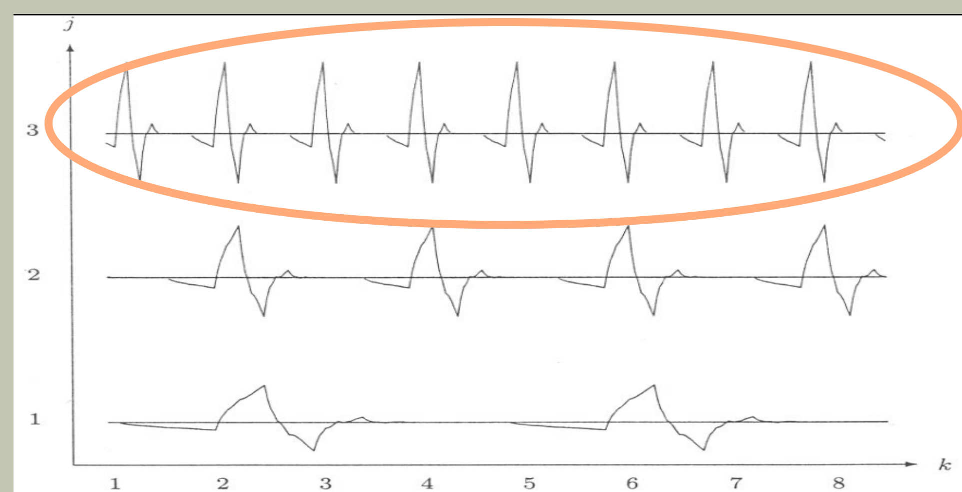




narrower, small translations

$$f(t) = \sum_{k} \sum_{j} a_{jk}\psi_{jk}(t)$$

fine details



j

coarse details

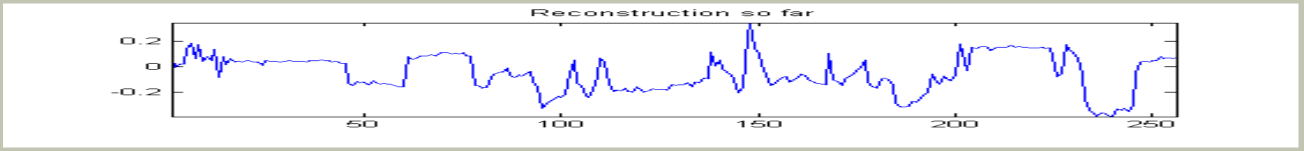
MULTIRESOLUTION REPRESENTATION USING

$\psi_{jk}(t)$

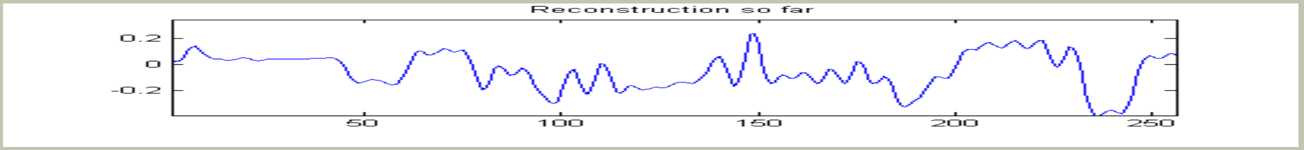
high resolutio

(more details)

$f(t)$

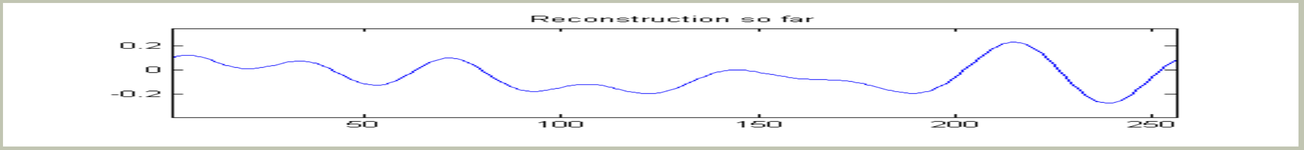


$\hat{f}_1(t)$



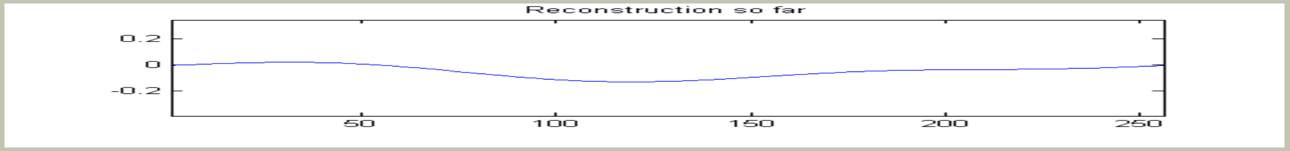
j

$\hat{f}_2(t)$



. . .

$\hat{f}_s(t)$



low resolutio

(less details)

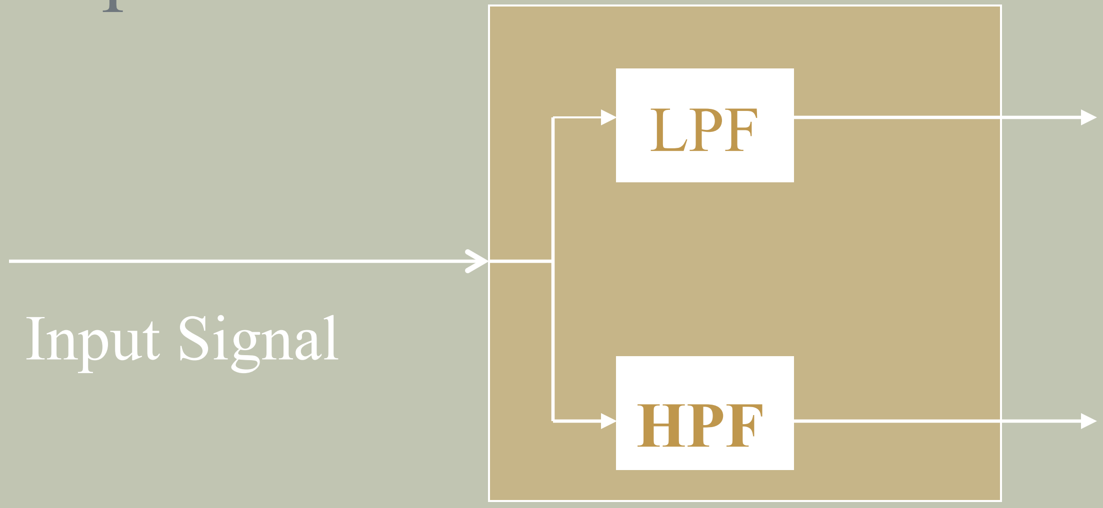
$$f(t) = \sum_k \sum_j a_{jk}\psi_{jk}(t)$$

Mallat* Filter Scheme

- Mallat was the first to implement this scheme, using a well known filter design called “two channel sub band coder”, yielding a *‘Fast Wavelet Transform’*

Approximations and Details:

- Approximations: High-scale, low-frequency components of the signal
- Details: low-scale, high-frequency components

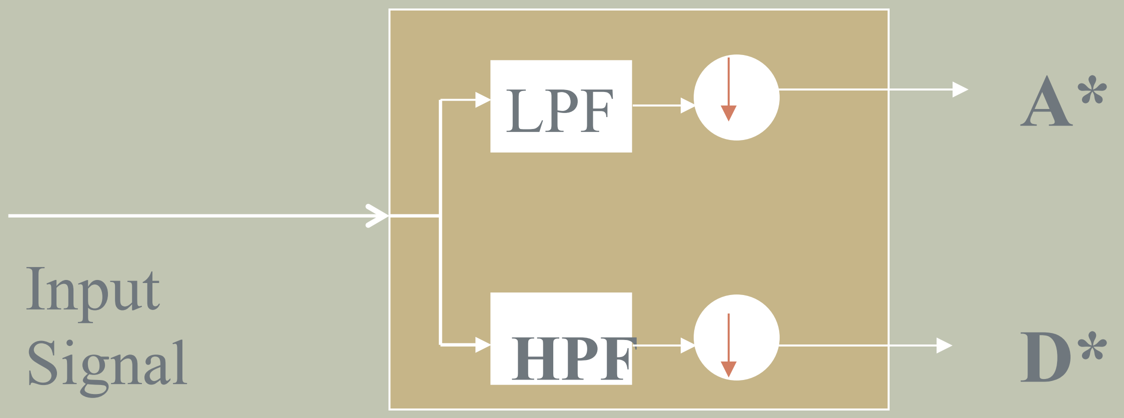


Decimation

- The former process produces twice the data it began with: N input samples produce N approximations coefficients and N detail coefficients.
- To correct this, we *Down sample (*or: *Decimate)* the filter output by two, by simply throwing away every second coefficient.

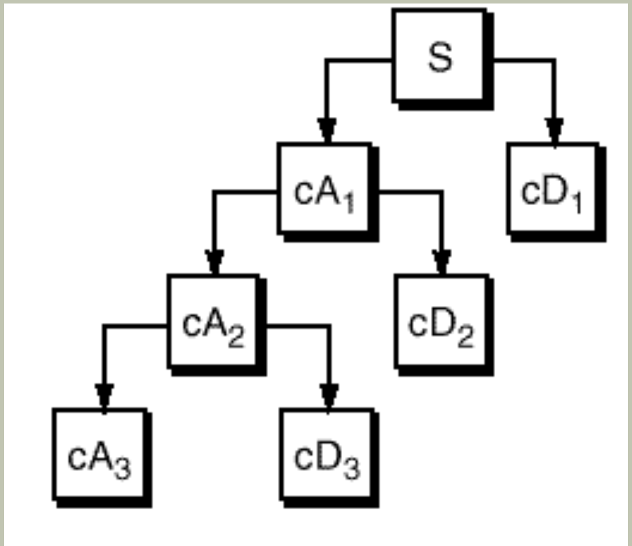
Decimation (cont'd)

So, a complete one stage block looks like:



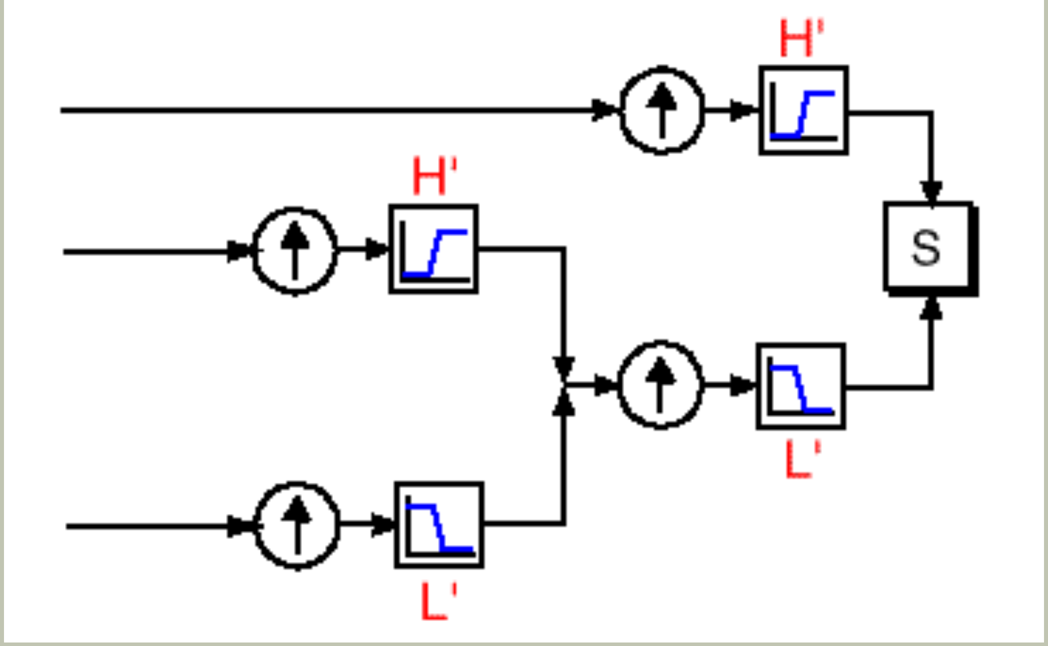
Multi-level Decomposition

- Iterating the decomposition process, breaks the input signal into many lower-resolution components: *Wavelet decomposition tree*:



Wavelet reconstruction

- Reconstruction (or synthesis) is the process in which we assemble all components back

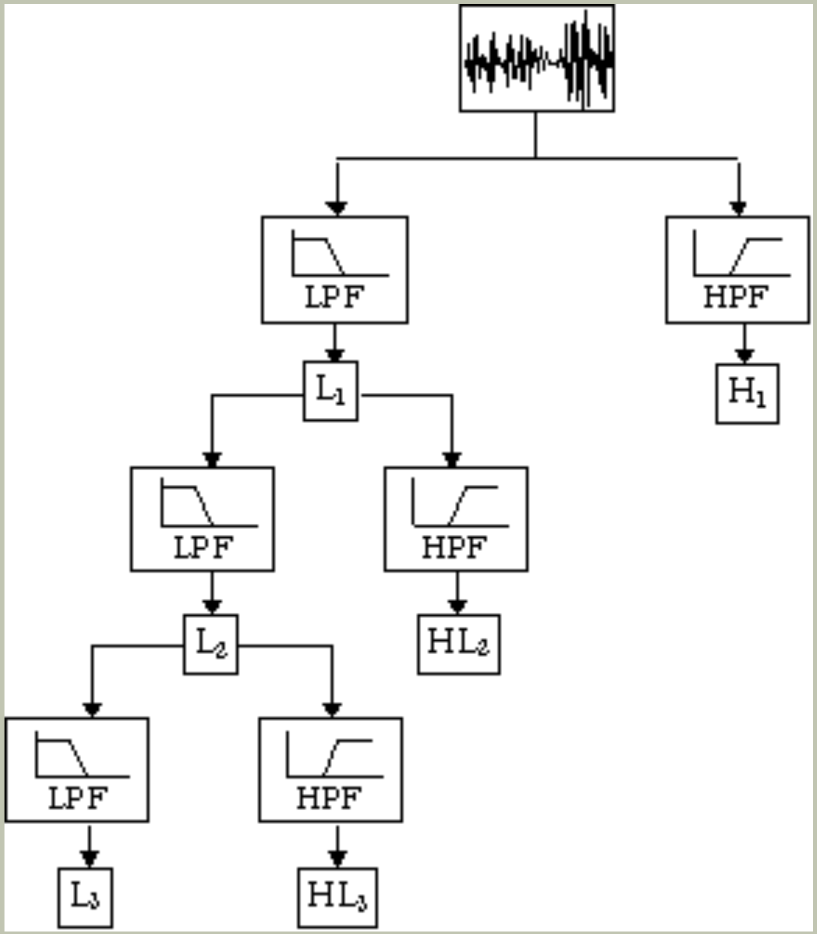


Up sampling (or interpolation) is done by zero inserting between every two coefficients

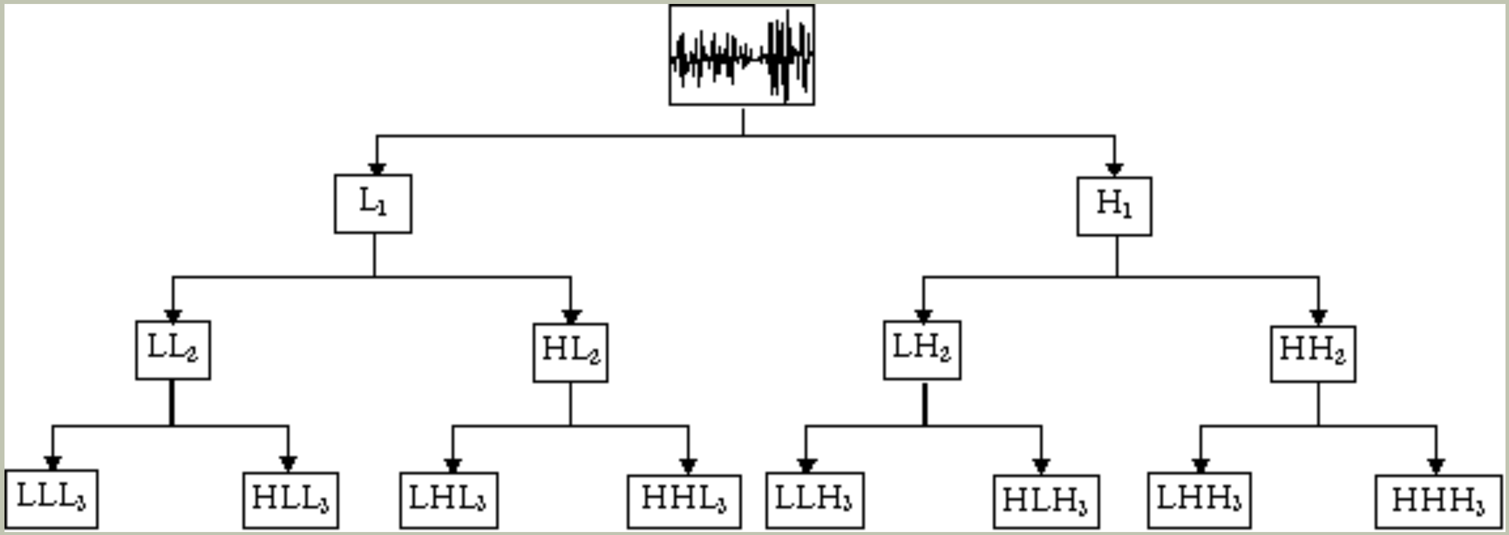
WAVELETS LIKE FILTERS

Relationship of Filters to Wavelet Shape

- Choosing the correct filter is most important.
- The choice of the filter determines the shape of the wavelet we use to perform the analysis.



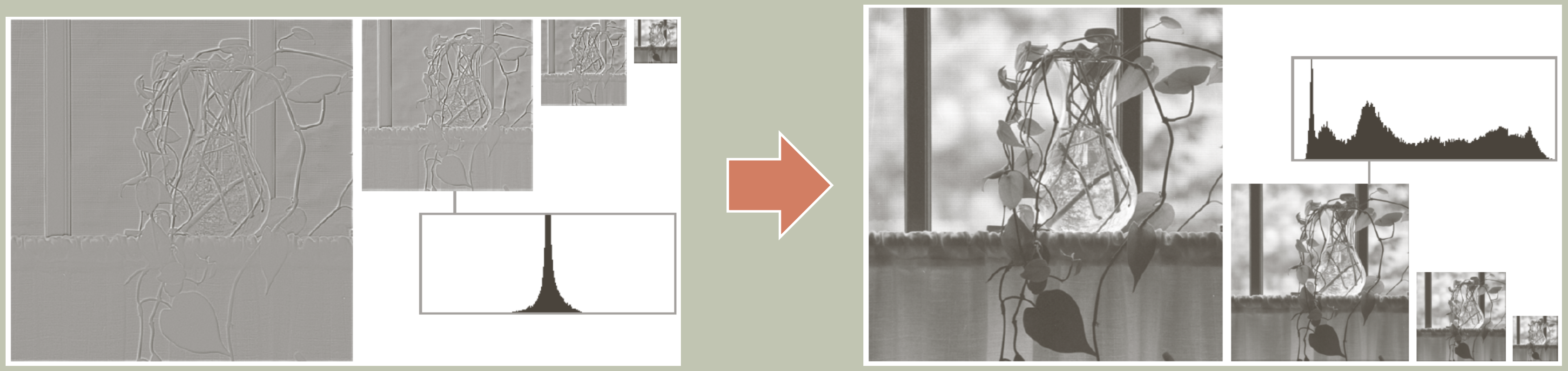
FILTER BANK REPRESENTATION OF THE DWT DILATIONS



WAVELET PACKET ANALYSIS (DWPA)
TREE DECOMPOSITION

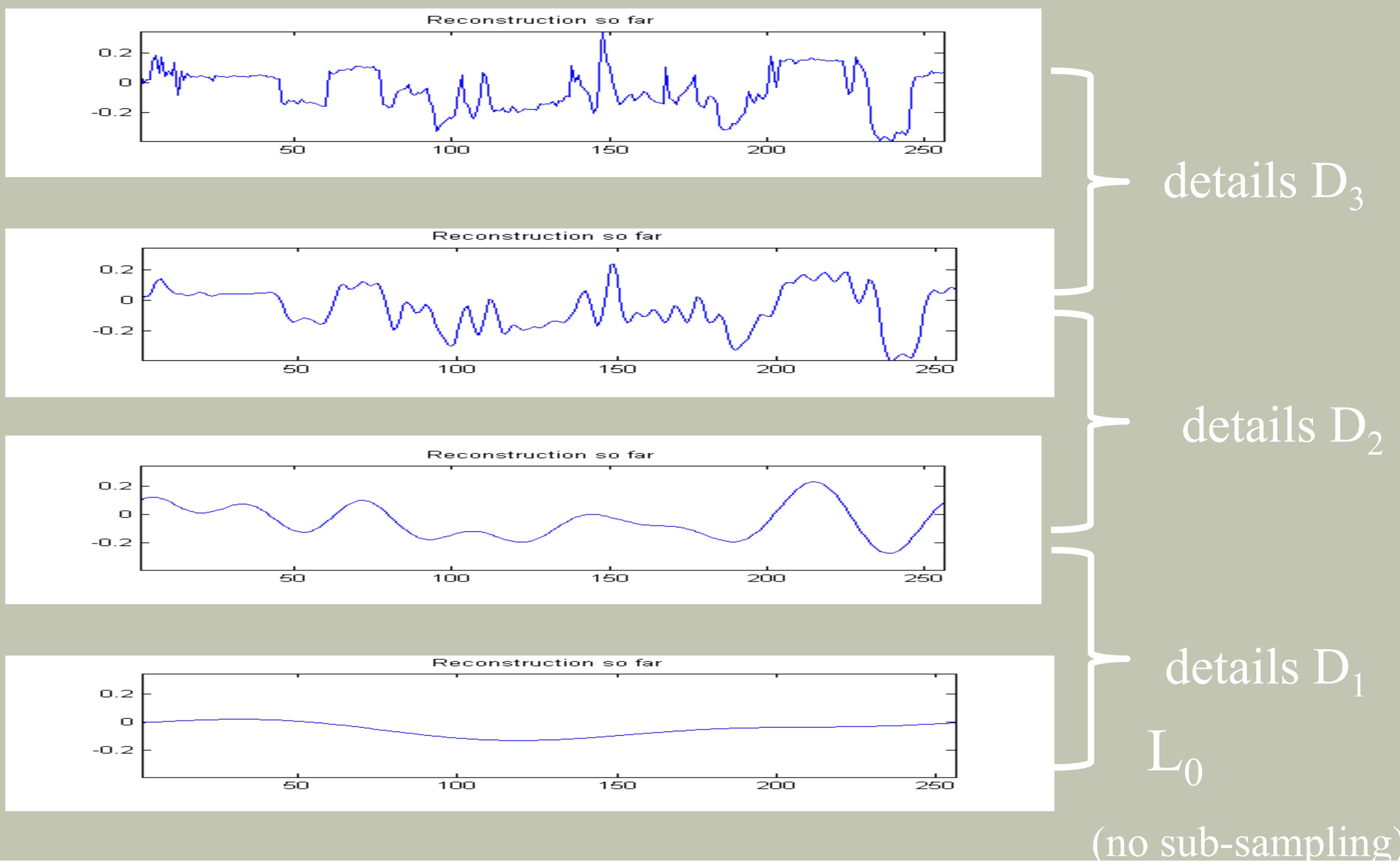
Prediction Residual Pyramid

- In the absence of quantization errors, the approximation pyramid can be reconstructed from the prediction residual pyramid.
- Prediction residual pyramid can be represented more efficiently.

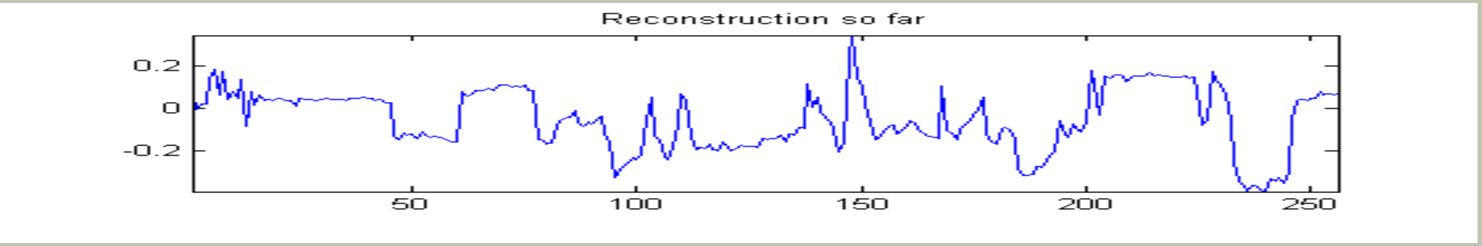


(with sub-sampling)

Efficient Representation Using "Details"



Efficient Representation Using Details (cont'd)



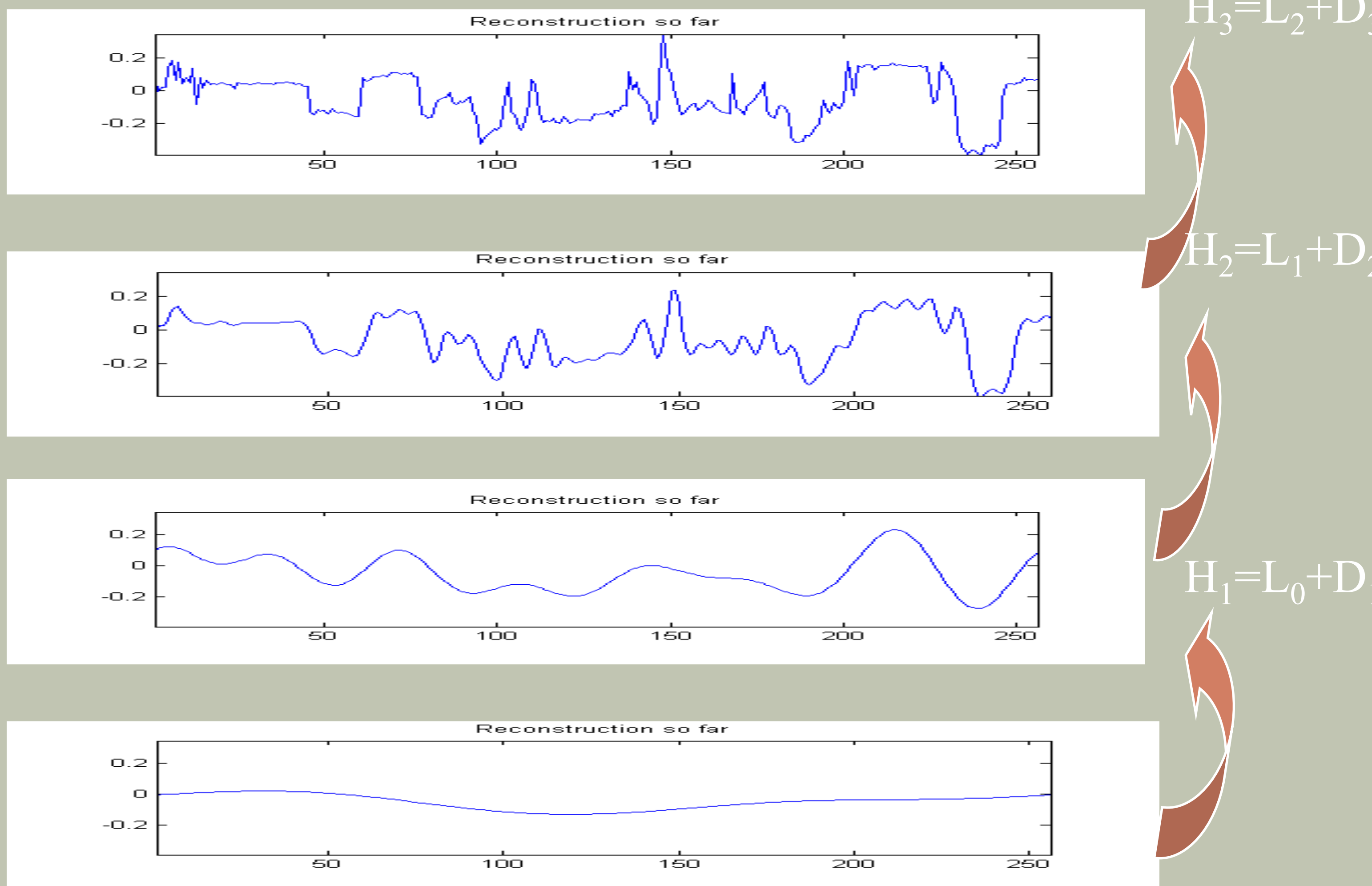
representation: $L_0\ D_1\ D_2\ D_3$

in general: $L_0\ D_1\ D_2\ D_3 \ldots D_J$

(decomposition or analysis)

A wavelet representation of a function consists of
(1) a coarse overall approximation
(2) detail coefficients that influence the function at various scales.

RECONSTRUCTION (SYNTHESIS)



details D_3

details D_2

details D_1

L_0

(no sub-sampling)

EXAMPLE - HAAR WAVELETS

- Suppose we are given a 1D "image" with a resolution of 4 pixels:

$$[9\ 7\ 3\ 5]$$

- The Haar wavelet transform is the following:

$$[6\ \ 2\ \ 1\ \ -1]$$

L_0 D_1 D_2 D_3

(with sub-sampling)

Example - Haar Wavelets (cont'd)

- Start by averaging the pixels together (pairwise) to get a new lower resolution image:

[8 4] (averaged and subsampled)

- To recover the original four pixels from the two averaged pixels, store some *detail coefficients.*

Resolution	*Averages*	*Detail Coefficients*
1	[9 7 3 5]	[]
2	[8 4]	[1 −1]

Example - Haar Wavelets (cont'd)

- Repeating this process on the averages gives the full decomposition:

Resolution	*Averages*	*Detail Coefficients*
1	[9 7 3 5]	[]
2	[8 4]	[1 −1]
4	[6]	[2]

Example - Haar Wavelets (cont'd)

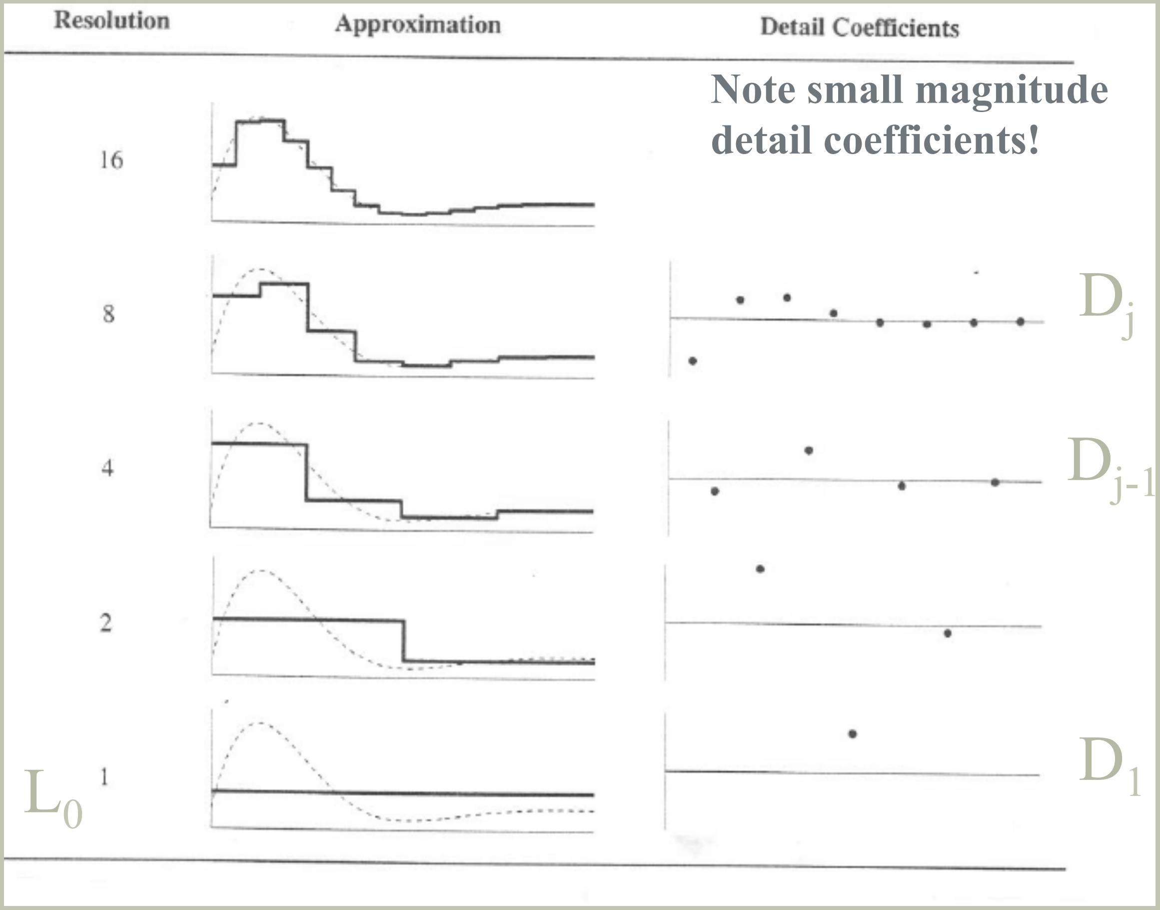
- The Harr decomposition of the original four-pixel image is:

$$[6 \quad 2 \quad 1 \quad -1]$$

- We can reconstruct the original image to a resolution by adding or subtracting the detail coefficients from the lower-resolution versions.

$[6 \quad 2 \quad 1 \quad -1]$ →(2) $[8 \quad 4]$ →(1 -1) $[9 \quad 7 \quad 3 \quad 5]$

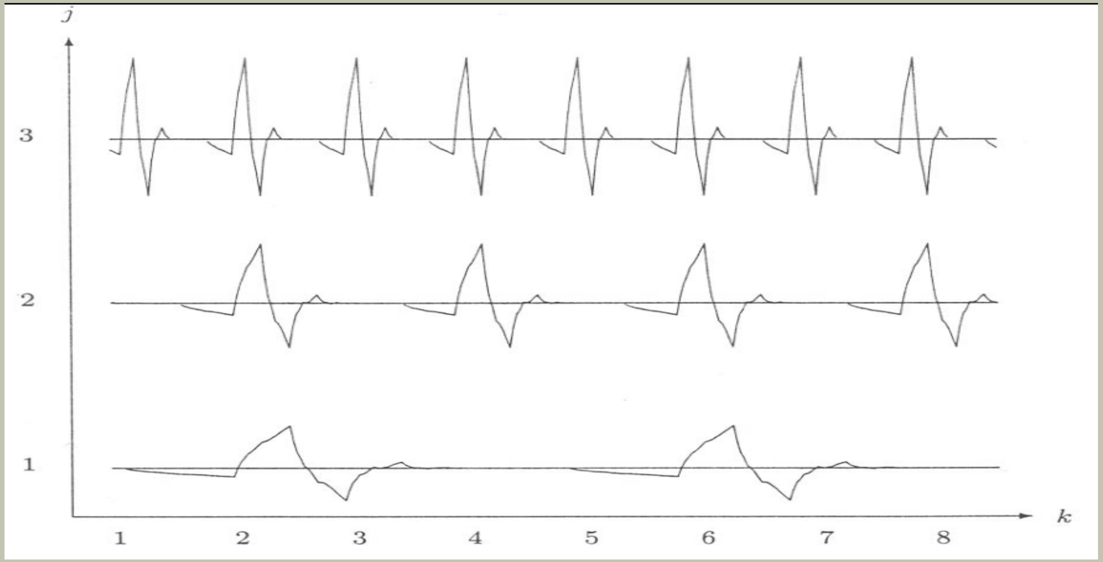
Example - Haar Wavelets (cont'd)



How to compute D_i ?

MULTIRESOLUTION CONDITIONS

- If a set of functions can be represented by a weighted sum of $\psi(2^j t - k)$, then a larger set, including the original, can be represented by a weighted sum of $\psi(2^{j+1} t - k)$:



scale/frequency localization

time localization

high resolution

j

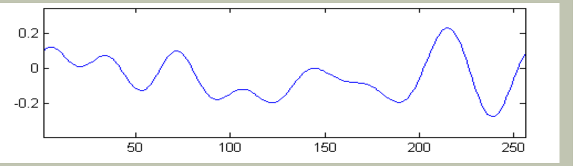
low resolution

Multiresolution Conditions (cont'd)

- If a set of functions can be represented by a weighted sum of $\psi(2^j t - k)$, then a larger set, including the original, can be represented by a weighted sum of $\psi(2^{j+1} t - k)$:

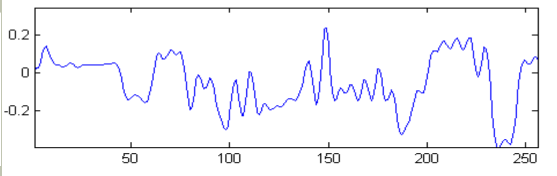
V_j: span of $\psi(2^j t - k)$:

$$f_j(t) = \sum_k a_k \psi_{jk}(t)$$



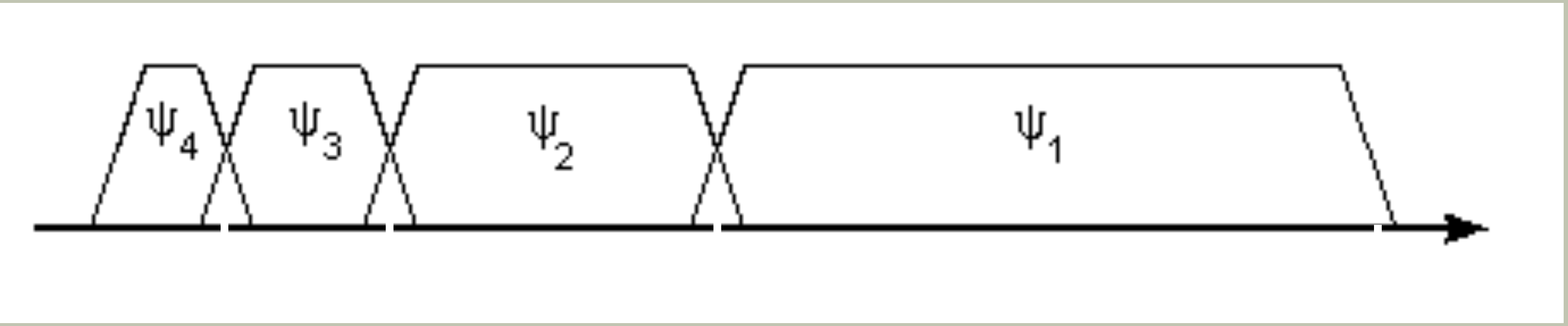
V_{j+1}: span of $\psi(2^{j+1} t - k)$:

$$f_{j+1}(t) = \sum_k b_k \psi_{(j+1)k}(t)$$



$$V_j \subseteq V_{j+1}$$

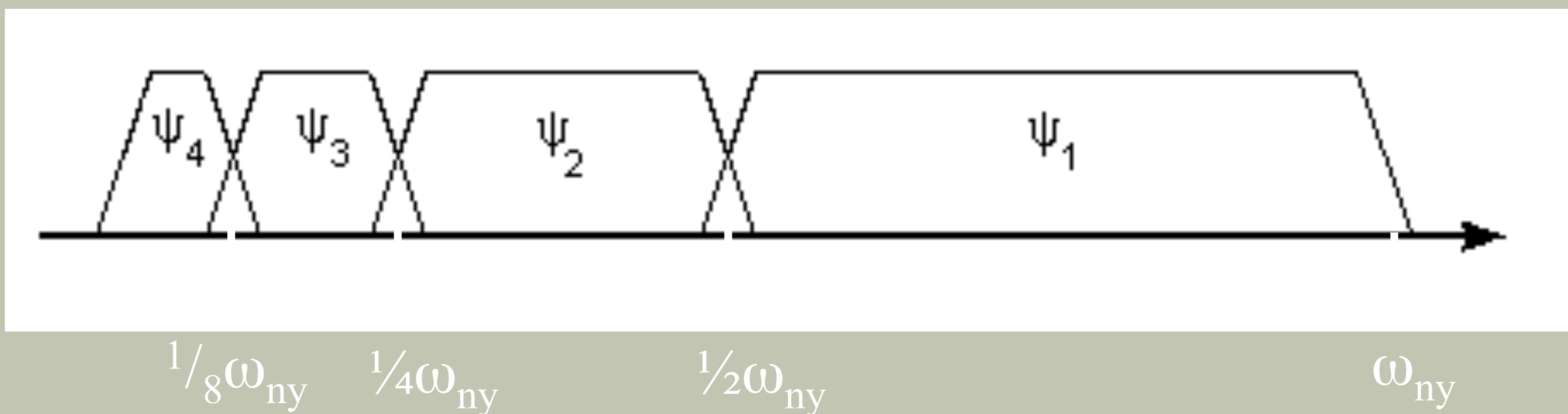
The factor of two scaling means that the spectra of the wavelets divide up the frequency scale into *octaves* (frequency doubling intervals)

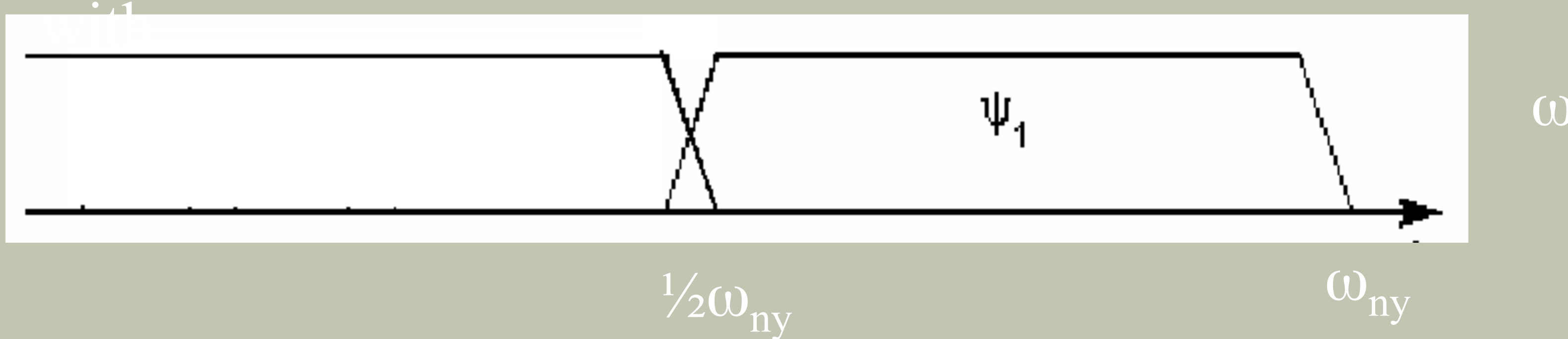


$^1/_8\omega_{ny}$ $\frac{1}{4}\omega_{ny}$ $\frac{1}{2}\omega_{ny}$ ω_{ny}

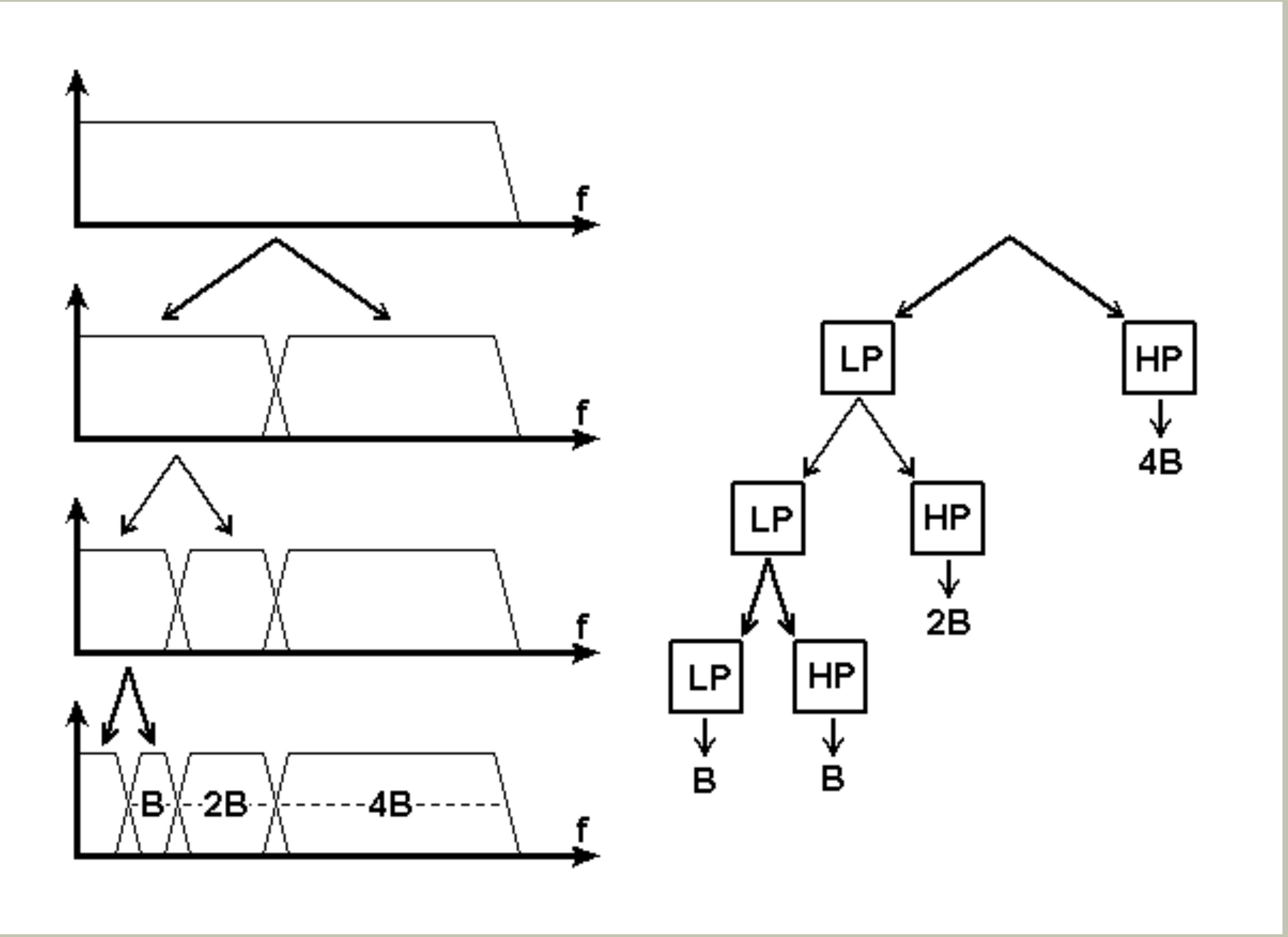
Ψ_1 is the wavelet, now viewed as a bandpass filter.

This suggests a recursion. Replace:





And then repeat the processes, recursively …



CHOOSING THE LOW-PASS FILTER

The low-pass filter, $f^{lp}(w)$ must match wavelet filter, $\Psi(\omega)$. A reasonable requirement is:

$$|f^{lp}(\omega)|^2 + |\Psi(\omega)|^2 = 1$$

That is, the spectra of the two filters add up to unity.

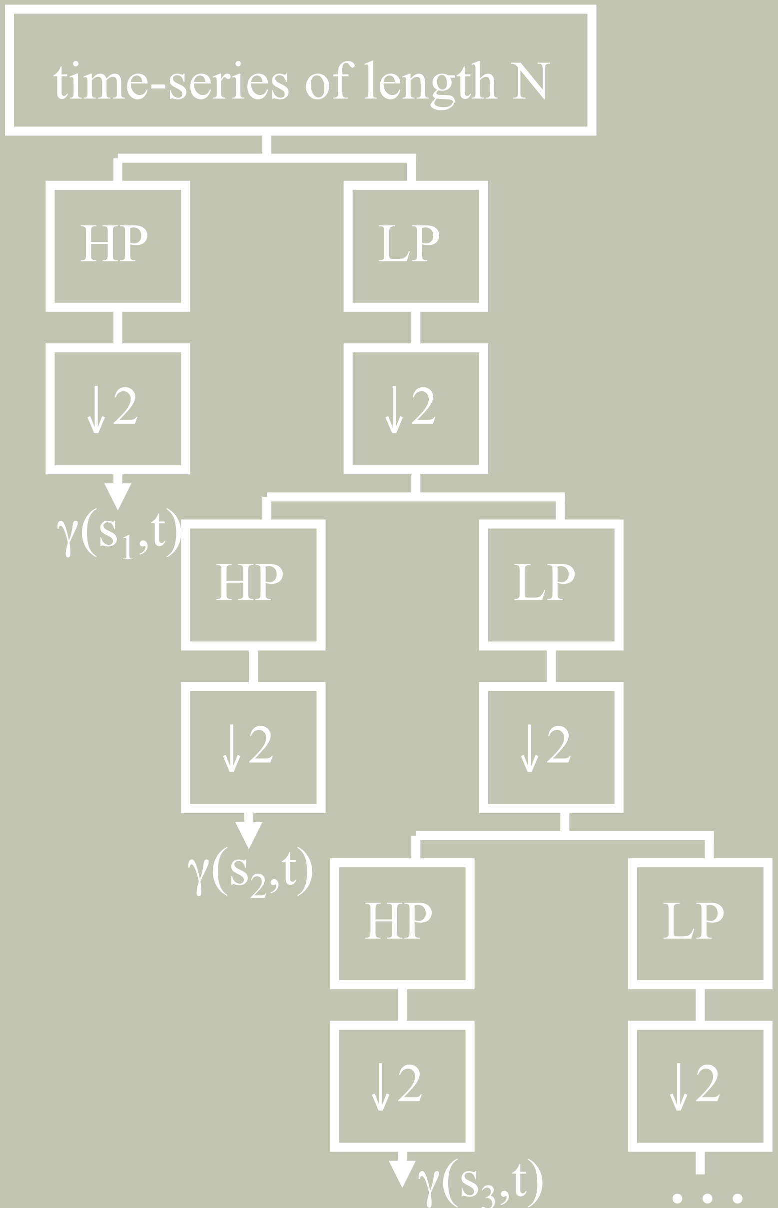
A pair of such filters are called *Quadature Mirror Filters*.

They are known to have filter coefficients that satisfy the relationship:

$$\Psi_{N-1-k} = (-1)^k f^{lp}{}_k$$

Furthermore, it's known that these filters allows perfect reconstruction of a time-series by summing its low-pass and high-pass versions

Recursion for wavelet coefficients

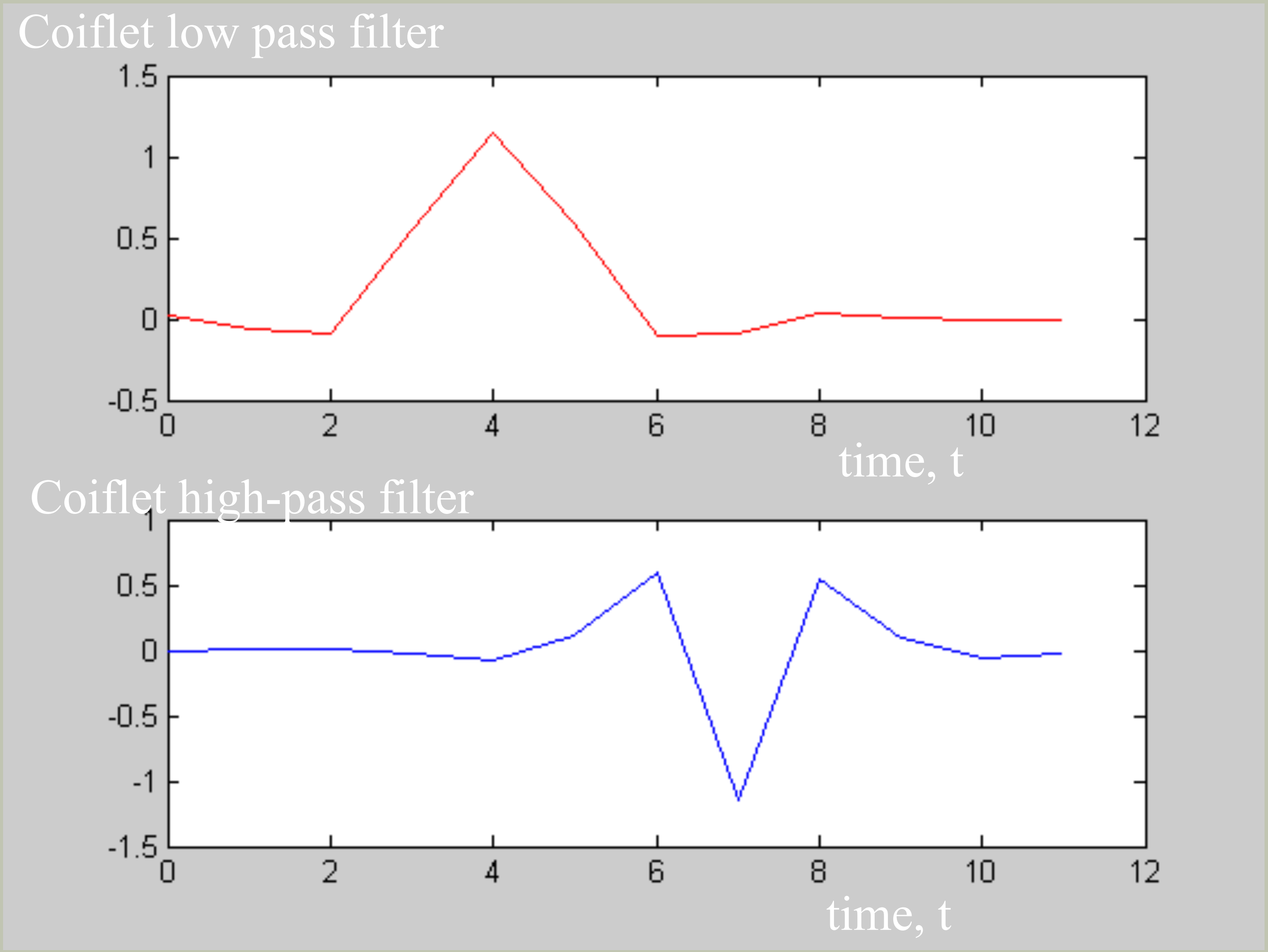


$\gamma(s_1,t)$: N/2 coefficients

$\gamma(s_2,t)$: N/4 coefficients

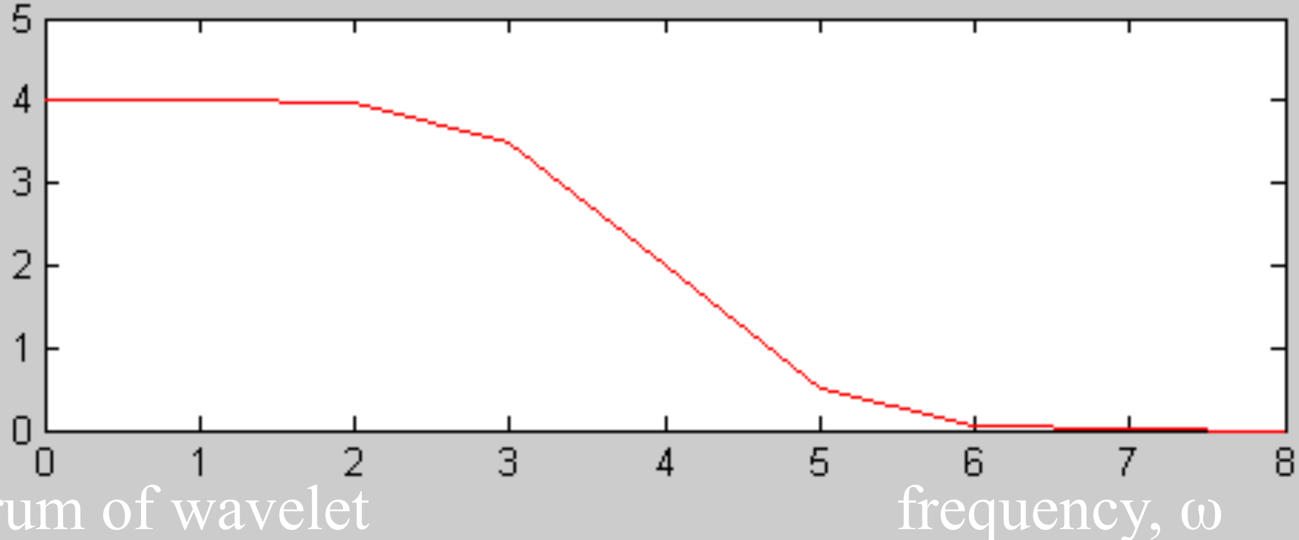
$\gamma(s_2,t)$: N/8 coefficients

Total: N coefficients

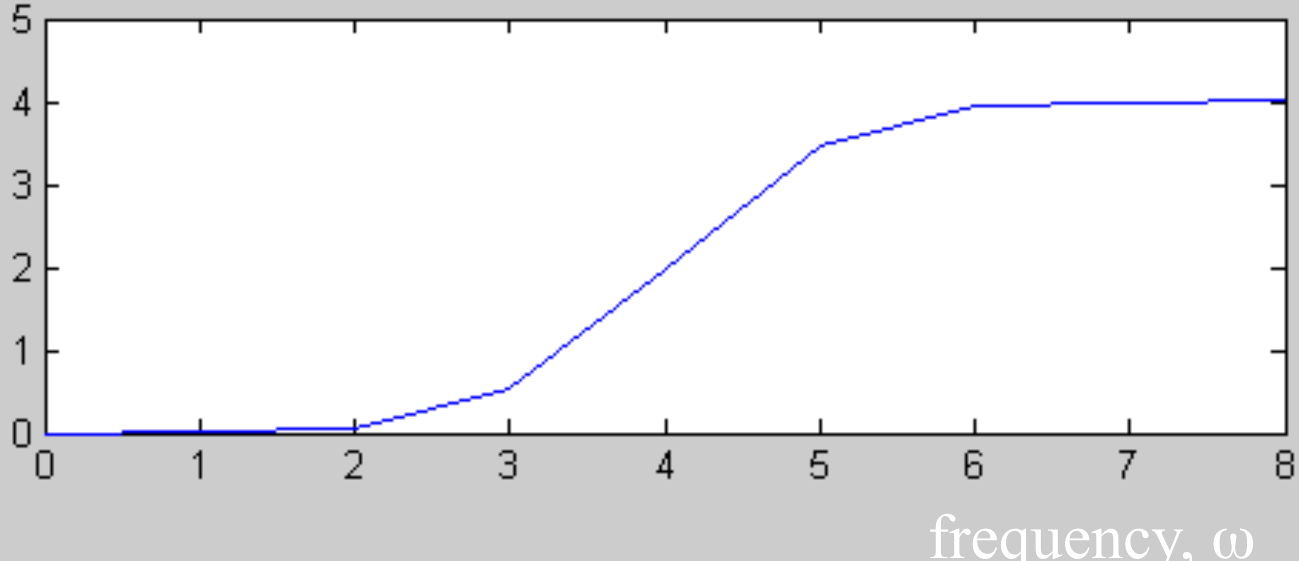


Coiflet low pass filter
1.5
1
0.5
0
-0.5
0
2
4
6
8
10
12
time, t
Coiflet high-pass filter
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
time, t

Spectrum of low pass filter



Spectrum of wavelet



SUMMARY: WAVELET EXPANSION

- Wavelet decompositions involve a pair of waveforms (mother wavelets):

encode low resolution info ← $\varphi(t)$ $\psi(t)$ → encode details or high resolution info

- The two shapes are *translated* and *scaled* to produce wavelets (wavelet basis) at different *locations* and on different *scales*.

$$\varphi(t-k) \quad \psi(2^j t-k)$$

Summary: wavelet expansion (cont'd)

- f(t) is written as a linear combination of φ(t-k) and $\psi(2^j t-k)$:

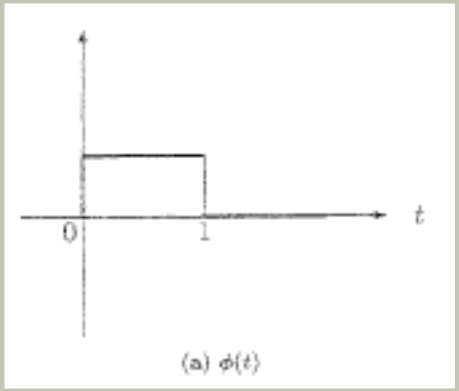
scaling function

wavelet function

1D HAAR WAVELETS

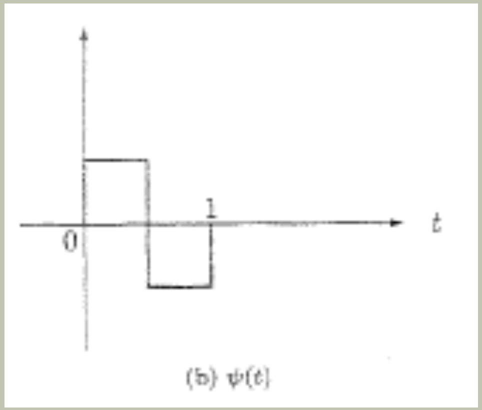
- Haar scaling and wavelet functions:

$\varphi(t)$



computes average

$\psi(t)$



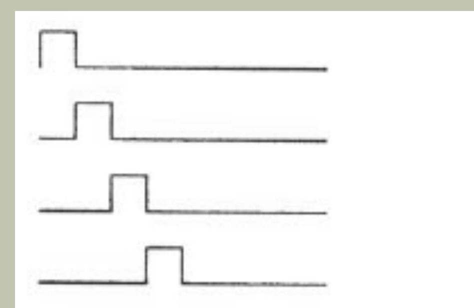
computes details

1D Haar Wavelets (cont'd)

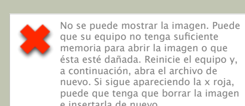
- V_j represents all the 2^j-pixel images
- Functions having constant pieces over 2^j equal-sized intervals on [0,1).

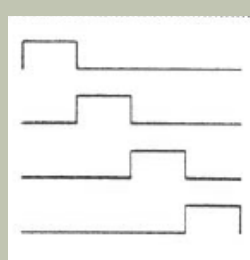
width: $1/2^j$

Examples:

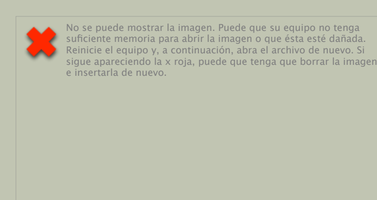


- Note that





$\in V_j$

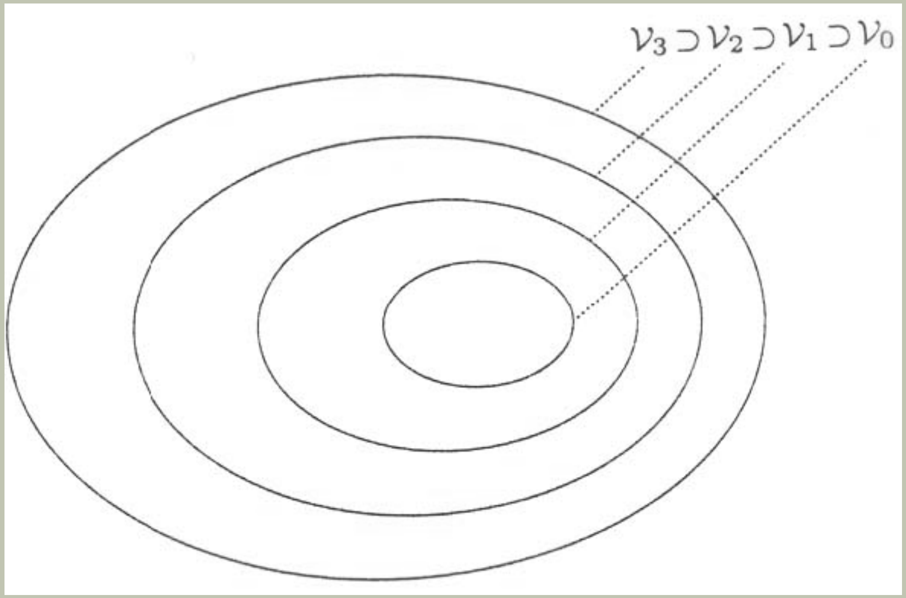


$\in V_j$

1D Haar Wavelets (cont'd)

$V_0, V_1, \ldots, V_j$ are nested

i.e., $V_j \subset V_{j+1}$



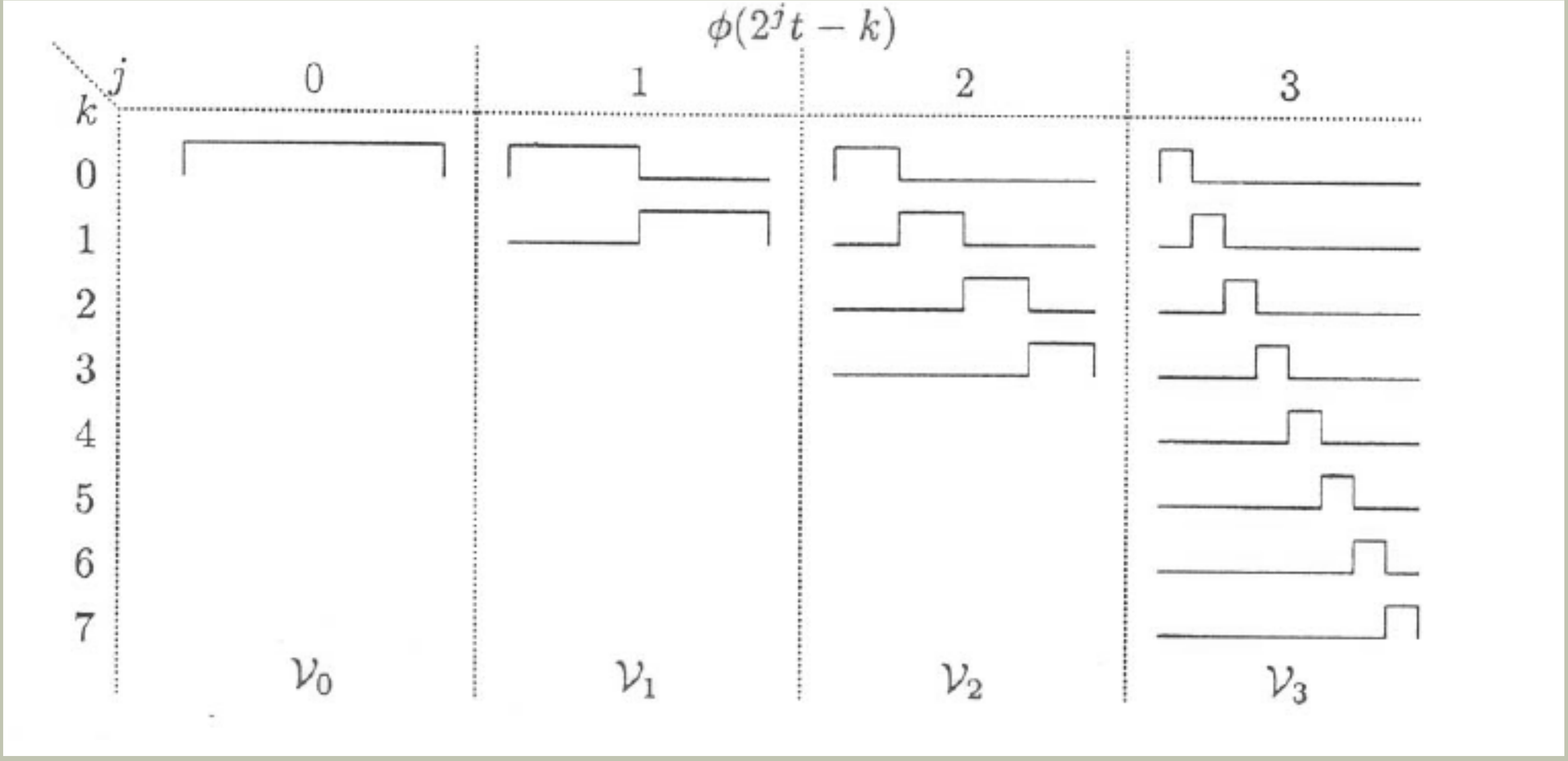
V_J fine details

…

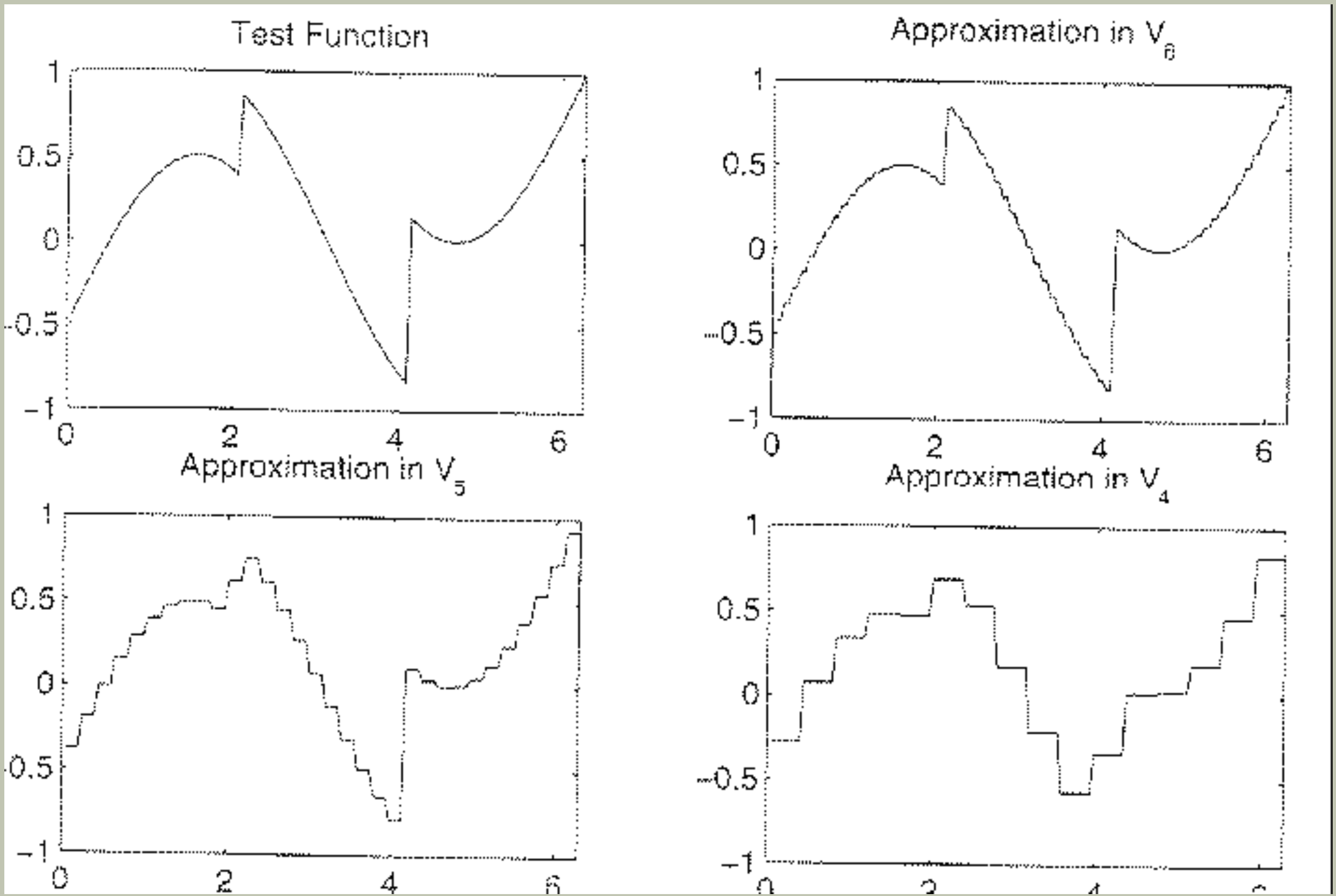
V_2

V_1 coarse details

1D Haar Wavelets (cont'd)

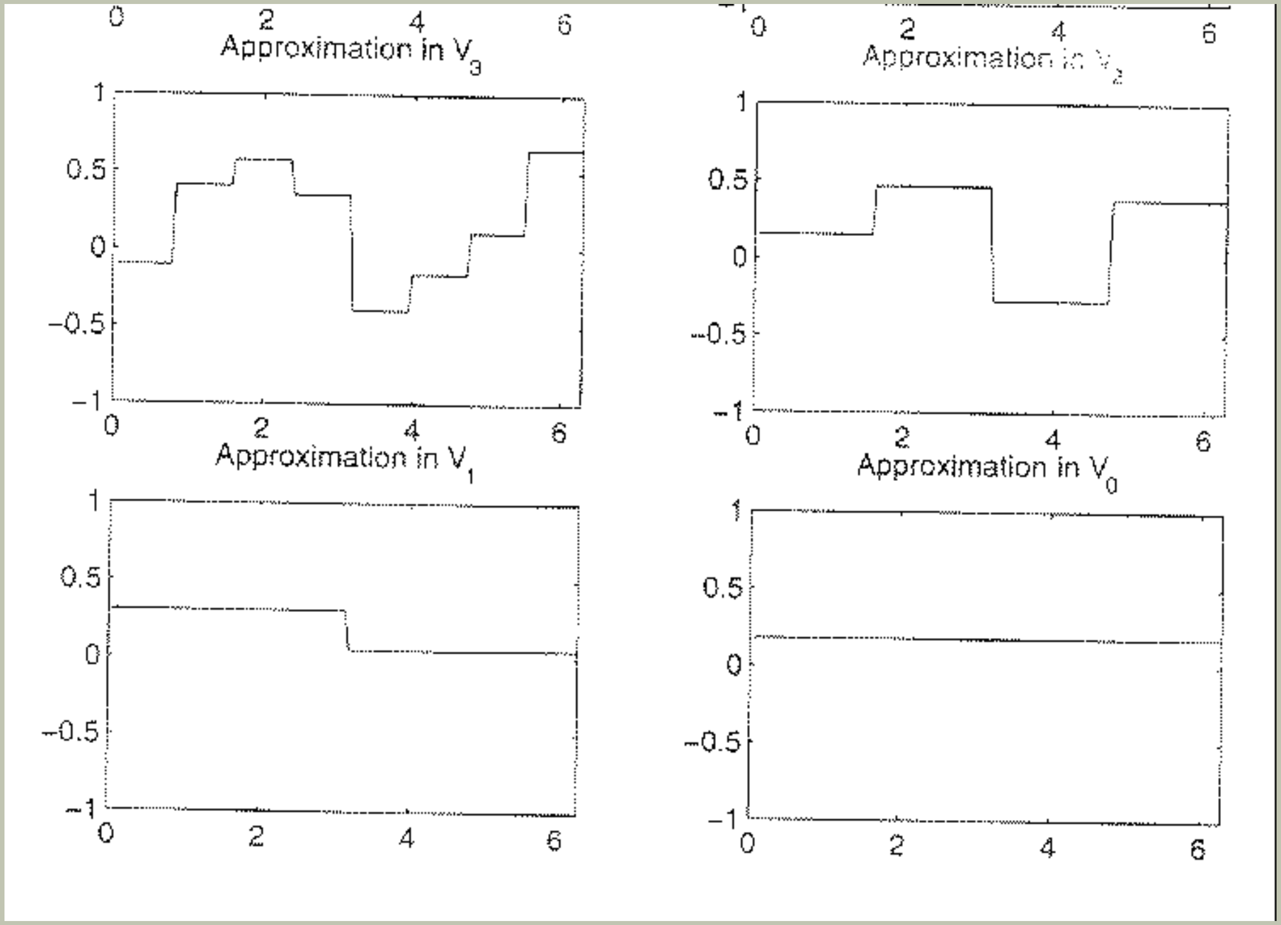


EXAMPLE



Test Function
Approximation in V_6
Approximation in V_5
Approximation in V_4
1
0.5
0
-0.5
-1
0
2
4
6

Example (cont'd)

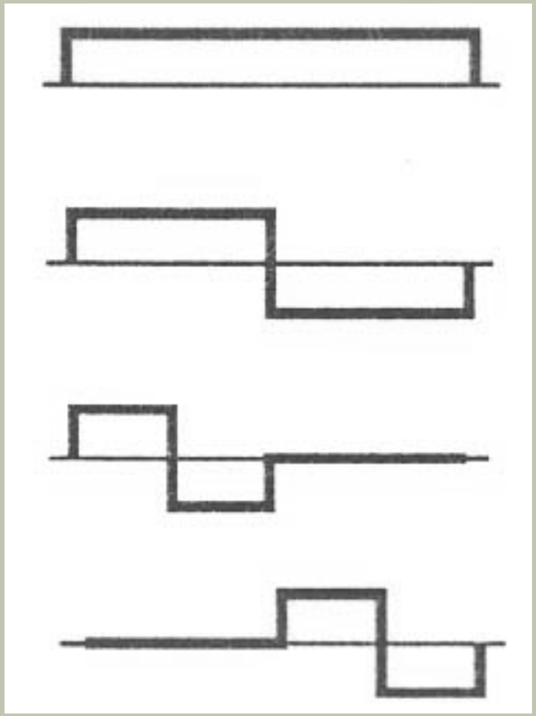


2D HAAR BASIS FOR STANDARD DECOMPOSITION

To construct the standard 2D Haar wavelet basis, consider all possible outer products of 1D basis functions.

Example:

$V_2 = V_0 + W_0 + W_1$



$\varphi_{0,0}(x)$

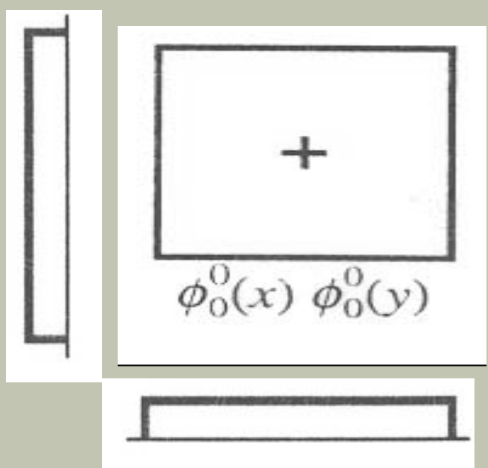
$\psi_{0,0}(x)$

$\psi_{0,1}(x)$

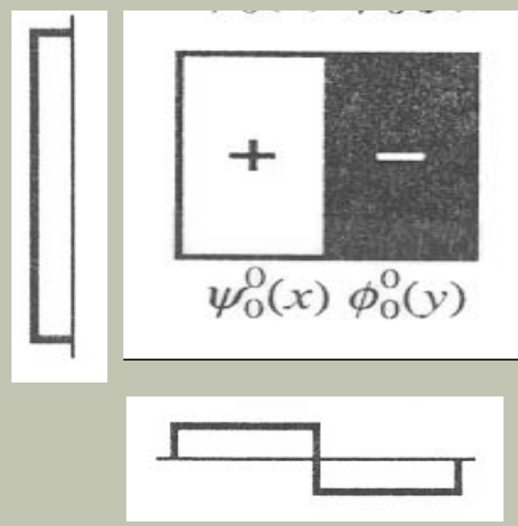
$\psi_{1,1}(x)$

2D HAAR BASIS FOR STANDARD DECOMPOSITION

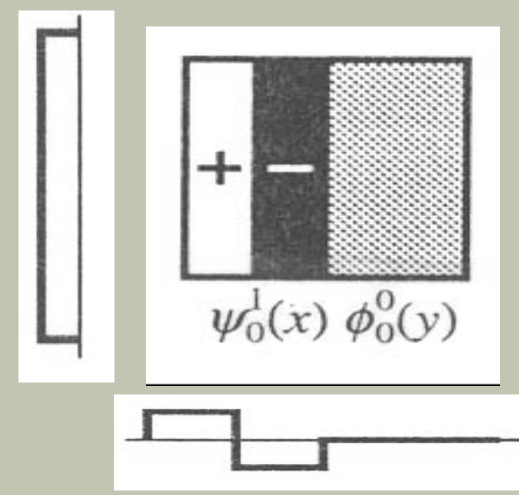
To construct the standard 2D Haar wavelet basis, consider all possible outer products of 1D basis functions.



$\varphi_{00}(x)$, $\varphi_{00}(x)$



$\psi_{00}(x)$, $\varphi_{00}(x)$

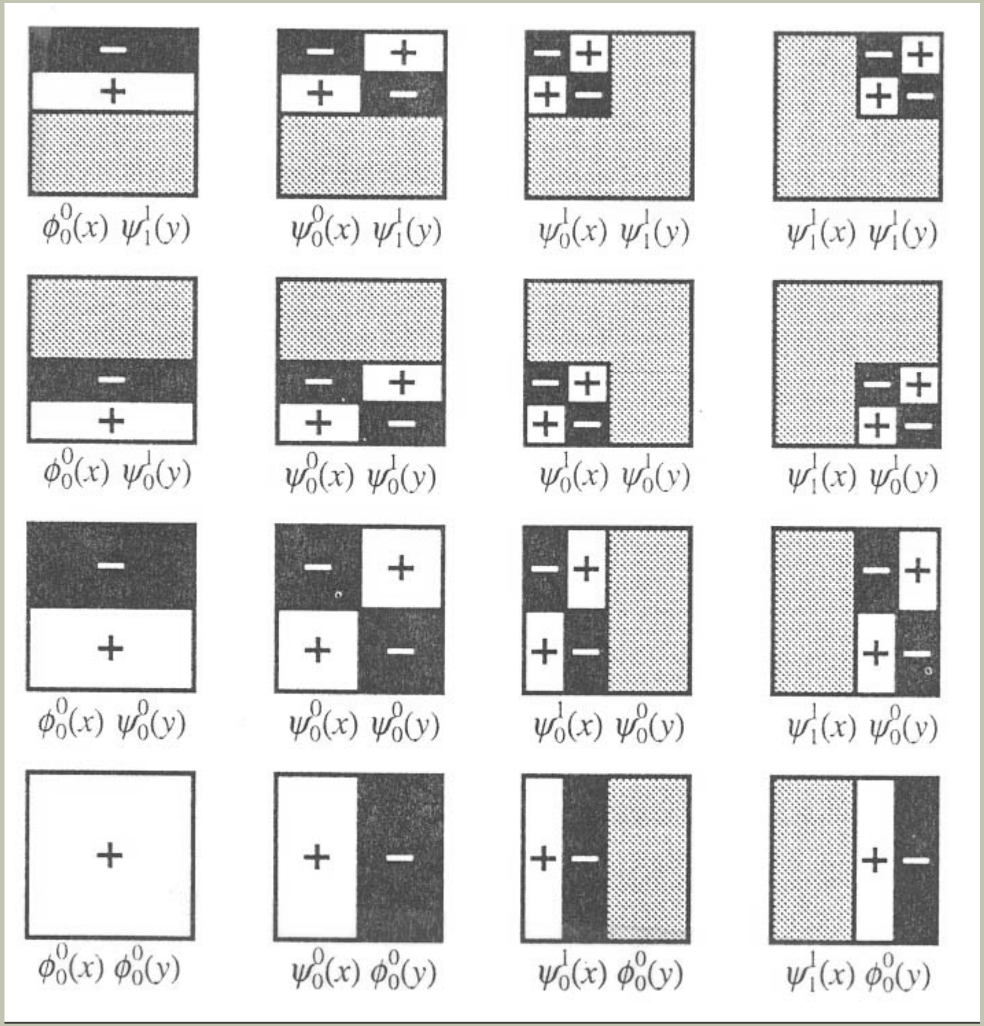


$\psi_{01}(x)$, $\varphi_{00}(x)$

$$\varphi_i^j(x) \equiv \varphi_{ji}(x) \qquad \psi_i^j(x) \equiv \psi_{ji}(x)$$

2D HAAR BASIS OF STANDARD DECOMPOSITION

V_2

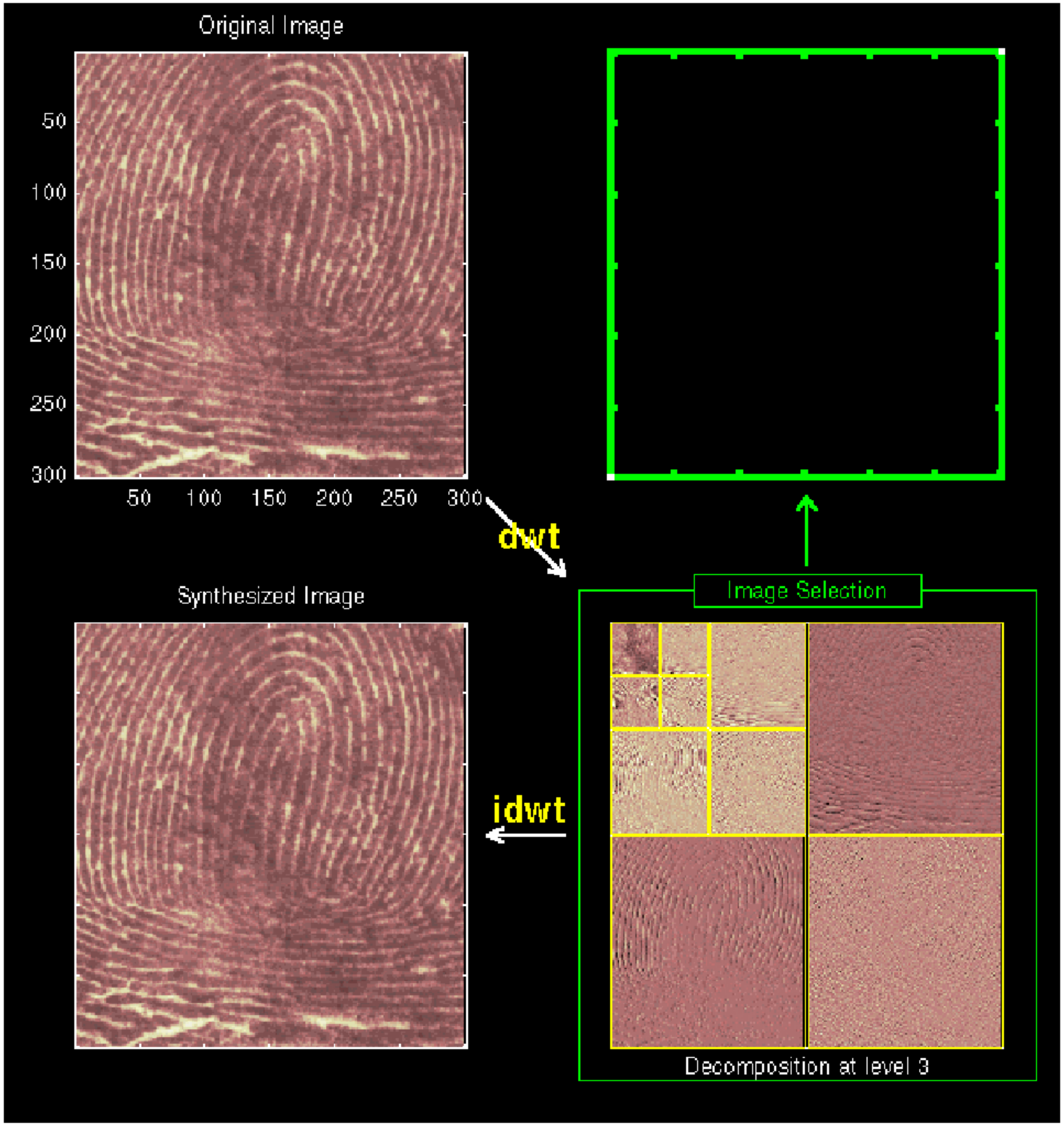


$$\varphi_i^j(x) \equiv \varphi_{ji}(x)$$

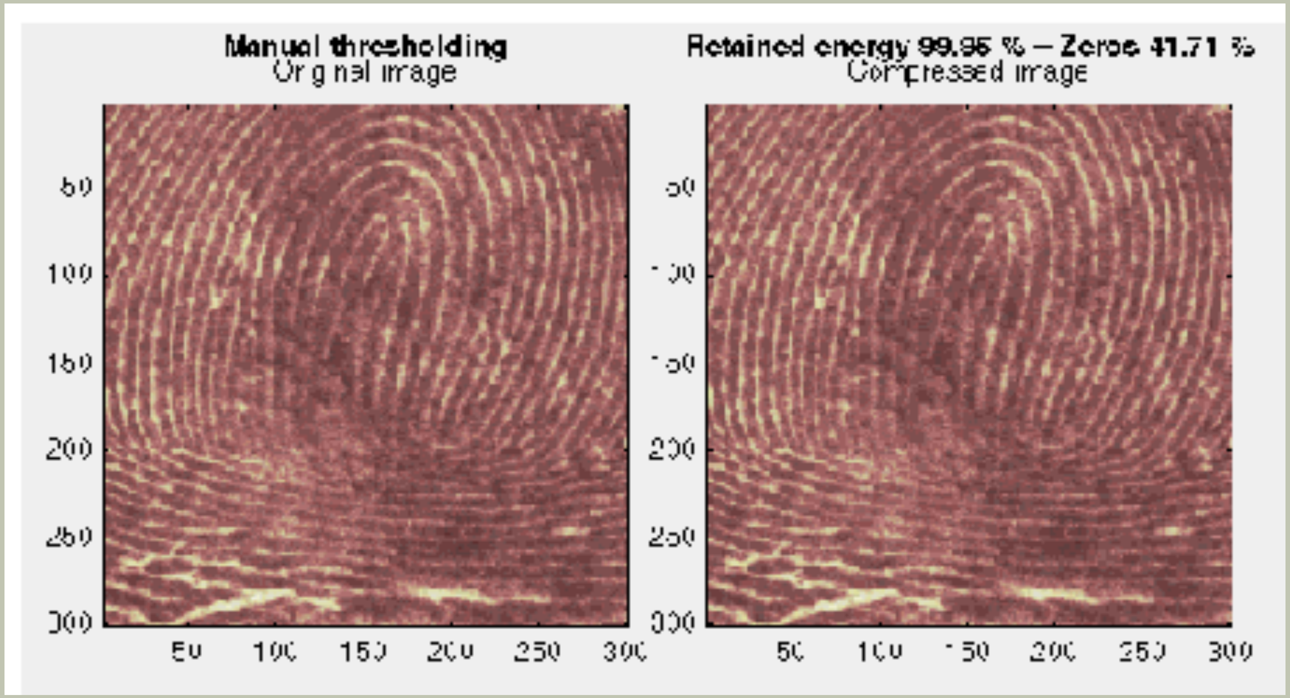
$$\psi_i^j(x) \equiv \psi_{ji}(x)$$

WAVELETS APPLICATIONS

- Noise filtering
- Image compression
- Fingerprint compression
- Image fusion
- Recognition
- Image matching and retrieval


Original Image
50
100
150
200
250
300
50
100
150
200
250
300
dwt
Image Selection
Synthesized Image
idwt
Decomposition at level 3

Original Image   Compressed Image



Threshold: 3.5
Zeros: 42%
Retained energy: 99.95%